VIII Quadratic Optimal Control Systems

VIII.1. Introduction

Note:
Performance index: a function whose value is used to indicate how well the actual performance of the system matches the desired performance.

Formulation of optimization problem:

- ✓ System equation
- ✓ Class of control vectors\constraints
- ✓ \performance index
- ✓ System parameter

Quadratic Optimal control
Let us consider the control system defined by

$$x((k+1)) = Gx(k) + Hu(k) \qquad 8.1$$

Where

$x(k)$ =n-vector
$u(k)$ =r-vector
$G = n \times n$ matrix
$H = n \times r$ matrix

In quadratic optimal control problem we desire to determine a law for the control vector $u(k)$ such that a given quadratic performance index is minimized.

An example of the quadratic performance index is

$$J = \frac{1}{2} x^*(N) S x(N) + \frac{1}{2} \sum_{k=0}^{N-1} \left[x^*(k) Q x(k) + u^*(k) R u(k) \right]$$

Where S, and Q are positive definite or positive semidefinite Hermitian matrices and R is a positive definite Hermitian matrix.

Notes:

1) Advantages of using the quadratic optimal control scheme is that the system designed will be asymptotically stable
2) We will use Langrange multipliers for optimal control problems
3) We will also use Liapunov approach.

VIII.2. Quadratic Optimal Control

$x(k+1) = Gx(k) + Hu(k)\,, \qquad x(0) = c$ 8.2

Where it is assumed to be completely controllable

$x(k)$=n-vector
$u(k)$=r-vector
$G = n \times n$ matrix
$H = n \times r$ matrix

Find the control vector $u(k)$ such that a given quadratic performance index is minimized. Performance index is

$$J = \frac{1}{2}x^*(N)Sx(N) + \frac{1}{2}\sum_{k=0}^{N-1}\left[x^*(k)Qx(k) + u^*(k)Ru(k)\right] \qquad 8.3$$

Where

Q is $n \times n$ positive definite or positive semidefinite Hermitian matrix (or real symmetric matrix)

R is $r \times r$ positive definite Hermitian matrix (or real symmetric matrix)

S is $n \times n$ positive definite or positive semidefinite Hermitian matrix (or real symmetric matrix)

Note: initial state of the system is at some arbitrary state $x(0) = c$. The final state may be fixed, in that case $\frac{1}{2}x^*(N)Sx(N)$ is removed from the performance index and final value of $x(N) = x_f$ is imposed.

Let us define the new performance index L using a set of Langrange multipliers:

$$L = \frac{1}{2}x^*(N)Sx(N) + \frac{1}{2}\sum_{k=0}^{N-1}\left\{\begin{matrix}\left[x^*(k)Qx(k) + u^*(k)Ru(k)\right] \\ + \lambda^*(k+1)\left[Gx(k) + Hu(k) - x(k+1)\right]\left[Gx(k) + Hu(k) - x(k+1)\right]^*\lambda(k+1)\end{matrix}\right\} \qquad 8.4$$

To minimize the function L, we need to differentiate L with respect to each component of vectors $x(k), u(k)$ and $\lambda(k)$ and set the results to equal to zero. However, it is convenient to differentiate L with respect $\bar{x}_i(k), \bar{u}_i(k)$ and $\bar{\lambda}_i(k)$ are, respectively, the complex conjugates of $x_i(k), u_i(k)$ and $\lambda_i(k)$.

$$\frac{\partial L}{\partial \bar{x}_i(k)} = 0,\; i = 1,2,\cdots n; k = 1,2,\cdots, N$$

$$\frac{\partial L}{\partial \overline{u}_i(k)} = 0,\ i = 1,2,\cdots,r; k = 0,1,2,\cdots,N-1$$
$$\frac{\partial L}{\partial \overline{\lambda}_i(k)} = 0,\ i = 1,2,\cdots n; k = 1,2,\cdots,N$$
Simplified expressions for the preceding partial derivative equations are
$$\frac{\partial L}{\partial \overline{x}(k)} = 0,\ k = 1,2,\cdots,N \qquad 8.5$$
$$\frac{\partial L}{\partial \overline{u}(k)} = 0,\ k = 0,1,2,\cdots,N-1 \qquad 8.6$$
$$\frac{\partial L}{\partial \overline{\lambda}(k)} = 0,\ k = 1,2,\cdots,N \qquad 8.7$$

Some formulas:
$$\frac{\partial}{\partial \overline{x}} x * Ax = Ax, \text{ and } \frac{\partial}{\partial \overline{x}} x * Ay = Ay,$$
Equation 8.5, 8.6, 8.7 may be obtained as follows:

$$\frac{\partial L}{\partial \overline{x}(k)} = 0,\ Qx(k) + G * \lambda(k+1) - \lambda(k) = 0, \quad k = 1,2,\cdots,N \qquad 8.8$$
$$\frac{\partial L}{\partial \overline{x}(N)} = 0,\ Sx(N) - \lambda(N) = 0, \quad k = 0,1,2,\cdots,N-1 \qquad 8.9$$
$$\frac{\partial L}{\partial \overline{u}(k)} = 0, \quad Ru(k) + H * \lambda(k+1) = 0, \quad k = 1,2,\cdots,N \qquad 8.10$$
$$\frac{\partial L}{\partial \overline{\lambda}(k)} = 0, \quad Gx(k-1) + Hu(k-1) - x(k) = 0, \quad k = 1,2,\cdots,N \qquad 8.11$$

Now from 8.8 we have

$$\lambda(k) = Qx(k) + G * \lambda(k+1), \quad k = 1,2,\cdots,N-1 \qquad 8.12$$
From 8.9 we have
$$\lambda(N) = Sx(N) \qquad 8.13$$

From 8.10
$$u(k) = -R^{-1}H * \lambda(k+1) \quad k = 0,1,2,\cdots,N-1 \qquad 8.14$$
From 8.11
$$x(k+1) = Gx(k) + Hu(k) \quad k = 0,1,2,\cdots,N-1 \qquad 8.15$$

Substitution of equation 8.14 into 8.15 results in

$$x(k+1) = Gx(k) - HR^{-1}H^*\lambda(k+1) \qquad x(0) = c \qquad 8.16$$

Next, we will obtain the optimal control vector $u(k)$ in the closed loop form by first obtaining the Riccati equation.

Assume $\lambda(k) = P(k)x(k)$ 8.17

Where $P(k)$ is an $n \times n$ Hermitian matrix (or an $n \times n$ real symmetric matrix).

Substitution of 8.17 into 8.12 results

$$P(k)x(k) = Qx(k) + G^*P(k+1)x(k+1), \qquad 8.18$$

Substitution of 8.17 into 8.16 results

$$x(k+1) = Gx(k) - HR^{-1}H^*P(k+1)x(k+1) \qquad 8.19$$

Remarks: the transformation process employed here is called the Riccati transformation.

From 8.19 we have $\left[I + HR^{-1}H^*P(k+1)\right]x(k+1) = Gx(k)$ 8.20

Inverse of $\left[I + HR^{-1}H^*P(k+1)\right]$ exists

Thus we have $x(k+1) = \left[I + HR^{-1}H^*P(k+1)\right]^{-1}Gx(k)$ 8.21

Substitution of 8.21 into 8.18, we obtain

$$P(k)x(k) = Qx(k) + G^*P(k+1)\left[I + HR^{-1}H^*P(k+1)\right]^{-1}Gx(k), \qquad 8.22$$

Or

$$\left(P(k) - Q - G^*P(k+1)\left[I + HR^{-1}H^*P(k+1)\right]^{-1}G\right)x(k) = 0 \qquad 8.23$$

Or

$$P(k) = Q - G^*P(k+1)\left[I + HR^{-1}H^*P(k+1)\right]^{-1}G \qquad 8.24$$

Or

$$P(k) = Q + G^*P(k+1)G - G^*P(k+1)H\left[R + H^*P(k+1)H\right]^{-1}H^*P(k+1)G \qquad 8.25$$

8.24 and 8.25 are called the Riccati equation.

From 8.13 and 8.17,

$$\lambda(N) = Sx(N) = P(N)x(N) \Rightarrow P(N) = S \qquad 8.26$$

Hence we can obtain $P(N)$, $P(N-1), \cdots P(0)$ starting from $P(N)$

From 8.14

$$
\begin{aligned}
u(k) &= -R^{-1}H^*\lambda(k+1) \\
&= -R^{-1}H^*(G^*)^{-1}[\lambda(k)-Qx(k)] \\
&= -R^{-1}H^*(G^*)^{-1}[P(k)-Q]x(k) \\
&= -K(k)x(k)
\end{aligned}
\qquad 8.27
$$

8.27 gives the closed loop form, or the feedback form, for the optimal control vector.

$$K(k) = R^{-1}H^*(G^*)^{-1}[P(k)-Q] \qquad 8.28$$

Note: $u(k)$ can be given in several different forms, which will results

$$K(k) = R^{-1}H^*\left[P^{-1}(k+1)+HR^{-1}H^*\right]^{-1}G \qquad 8.29$$

Or

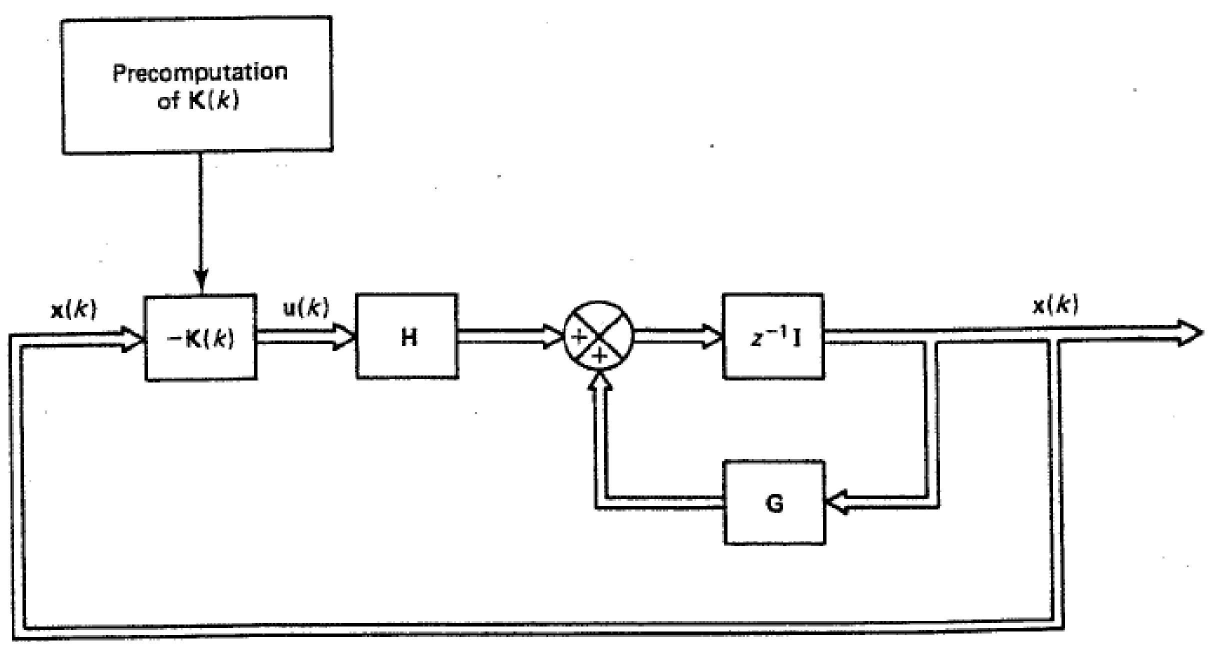
$$K(k) = \left[R+H^*P(k+1)H\right]^{-1}H^*P(k+1)G \qquad 8.30$$

Evaluation of the minimum performance index

$$\min J = \min\left\{\frac{1}{2}x^*(N)Sx(N)+\frac{1}{2}\sum_{k=0}^{N-1}[x^*(k)Qx(k)+u^*(k)Ru(k)]\right\}$$

Premultiplying 8.18 both side by $x^*(k)$

$$x^*(k)P(k)x(k) = x^*(k)Qx(k)+x^*(k)G^*P(k+1)x(k+1),$$



Optimal regular system based on a quadratic performance index

From 8-20 we have Substitution 8-20 into this equation

$$\left[I + HR^{-1}H^* P(k+1)\right]x(k+1) = Gx(k) \Rightarrow x^*(k)G^* = x^*(k+1)\left[I + HR^{-1}H^* P(k+1)\right]^*$$

$$\begin{aligned} x^*(k)P(k)x(k) &= x^*(k)Qx(k) + x^*(k+1)\left[I + HR^{-1}H^* P(k+1)\right]^* P(k+1)x(k+1), \\ &= x^*(k)Qx(k) + x^*(k+1)\left[I + P(k+1)HR^{-1}H^*\right]P(k+1)x(k+1), \end{aligned} \quad 8.31$$

Hence

$$\begin{aligned} & x^*(k)Qx(k) = x^*(k)P(k)x(k) - x^*(k+1)P(k+1)x(k+1) \\ & - x^*(k+1)P(k+1)HR^{-1}H^* P(k+1)x(k+1) \end{aligned} \quad 8.32$$

From 8.14, 8.17

$$u(k) = -R^{-1}H^*\lambda(k+1) = -R^{-1}H^* P(k+1)x(k+1)$$

$$\begin{aligned} u^*(k)Ru(k) &= -x^*(k+1)P(k+1)HR^{-1}R\left[-R^{-1}H^* P(k+1)x(k+1)\right] \\ &= x^*(k+1)P(k+1)HR^{-1}H^* P(k+1)x(k+1) \end{aligned} \quad 8.33$$

Adding 8.32 and 8.33 together:

$$x^*(k)Qx(k) + u^*(k)Ru(k) = x^*(k)P(k)x(k) - x^*(k+1)P(k+1)x(k+1) \quad 8.34$$

By substituting 8.34 into 8.3, we will get 8.35

$$\begin{aligned} J_{\min} &= \frac{1}{2}\mathbf{x}^*(N)\mathbf{S}\mathbf{x}(N) + \frac{1}{2}\sum_{k=0}^{N-1}\left[\mathbf{x}^*(k)\mathbf{P}(k)\mathbf{x}(k) - \mathbf{x}^*(k+1)\mathbf{P}(k+1)\mathbf{x}(k+1)\right] \\ &= \frac{1}{2}\mathbf{x}^*(N)\mathbf{S}\mathbf{x}(N) + \frac{1}{2}[\mathbf{x}^*(0)\mathbf{P}(0)\mathbf{x}(0) - \mathbf{x}^*(1)\mathbf{P}(1)\mathbf{x}(1) + \mathbf{x}^*(1)\mathbf{P}(1)\mathbf{x}(1) \\ &\quad - \mathbf{x}^*(2)\mathbf{P}(2)\mathbf{x}(2) + \cdots + \mathbf{x}^*(N-1)\mathbf{P}(N-1)\mathbf{x}(N-1) - \mathbf{x}^*(N)\mathbf{P}(N)\mathbf{x}(N)] \\ &= \frac{1}{2}\mathbf{x}^*(N)\mathbf{S}\mathbf{x}(N) + \frac{1}{2}\mathbf{x}^*(0)\mathbf{P}(0)\mathbf{x}(0) - \frac{1}{2}\mathbf{x}^*(N)\mathbf{P}(N)\mathbf{x}(N) \end{aligned} \quad (8\text{–}35)$$

Since we have $P(N) = S$,

8.35 will become:

$$J_{\min} = x^*(0)P(0)x(0) \quad 8.36$$

Thus the minimum value of the performance index J is given by equation 8.36. It is a function of $P(0)$ and the initial state $x(0)$.

Example 8.1 consider the discrete time system

$$x(k+1) = Gx(k) + Hu(k)$$

Where

$$G = \begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Determine the optimal control sequence $u(k)$ that will minimize the following performance index:

$$J = \left\{ \frac{1}{2} x^*(8) S x(8) + \frac{1}{2} \sum_{k=0}^{7} \left[x^*(k) Q x(k) + u^*(k) R u(k) \right] \right\}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1 \quad S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P(k) = Q + G^* P(k+1) \left[I + HR^{-1} H^* P(k+1) \right]^{-1} G$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p_{11}(k+1) & p_{12}(k+1) \\ p_{12}(k+1) & p_{22}(k+1) \end{bmatrix}$$

$$\times \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p_{11}(k+1) & p_{12}(k+1) \\ p_{12}(k+1) & p_{22}(k+1) \end{bmatrix} \right\}^{-1} \begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix}$$

The boundary condition for $P(k)$ is

$$P(8) = S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence

$$P(7) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}^{-1}$$

$$\times \begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix} = \begin{bmatrix} 1.1667 & -0.1667 \\ -0.1667 & 1.6667 \end{bmatrix}$$

$$P(6) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1.1667 & -0.1667 \\ -0.1667 & 1.6667 \end{bmatrix}$$

$$\times \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1.1667 & -0.1667 \\ -0.1667 & 1.6667 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.2560 & -0.2143 \\ -0.2143 & 1.7143 \end{bmatrix}$$

Similarly, we can obtain $P(5)$, $P(4)$, ..., $P(0)$ as shown in Table B-8-1.

Next we determine the feedback gain matrix $K(k)$. Referring to Equation 28, we have

$$K(k) = R^{-1} H^* (G^*)^{-1} \left[P(k) - Q \right]$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} p_{11}(k) & p_{12}(k) \\ p_{12}(k) & p_{22}(k) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} p_{11}(k) - 1 & p_{12}(k) \\ p_{12}(k) & p_{22}(k) - 1 \end{bmatrix}$$

$$= \begin{bmatrix} p_{12}(k) & p_{22}(k) - 1 \end{bmatrix}$$

Hence, we find

$$\underset{\sim}{K}(8) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{K}(7) = \begin{bmatrix} -0.1667 & 0.6667 \end{bmatrix}$$

$$\underset{\sim}{K}(6) = \begin{bmatrix} -0.2143 & 0.7143 \end{bmatrix}$$

Similarly, we get $\underset{\sim}{K}(5)$, $\underset{\sim}{K}(4)$, ..., $\underset{\sim}{K}(0)$ as given in Table B-8-1.

To compute $\underset{\sim}{x}(k)$, first define

$$\underset{\sim}{K}(k) = \begin{bmatrix} K_1(k) & K_2(k) \end{bmatrix}$$

Then

$$u(k) = -\underset{\sim}{K}(k)\underset{\sim}{x}(k) = -\begin{bmatrix} K_1(k) & K_2(k) \end{bmatrix} \underset{\sim}{x}(k)$$

and

$$\underset{\sim}{x}(k+1) = \underset{\sim}{G}\underset{\sim}{x}(k) + \underset{\sim}{H}u(k) = \begin{bmatrix} \underset{\sim}{G} - \underset{\sim}{H}\underset{\sim}{K}(k) \end{bmatrix} \underset{\sim}{x}(k)$$

$$= \begin{bmatrix} -K_1(k) & 1 - K_2(k) \\ -0.5 - K_1(k) & 1 - K_2(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Hence

$$\underset{\sim}{x}(1) = \begin{bmatrix} 0.2114 & 0.2803 \\ -0.2886 & 0.2803 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.9834 \\ -0.0166 \end{bmatrix}$$

$$\underset{\sim}{x}(2) = \begin{bmatrix} 0.2114 & 0.2803 \\ -0.2886 & 0.2803 \end{bmatrix} \begin{bmatrix} 0.9834 \\ -0.0166 \end{bmatrix} = \begin{bmatrix} 0.2032 \\ -0.2885 \end{bmatrix}$$

Similarly, we can obtain $\underset{\sim}{x}(3)$, $\underset{\sim}{x}(4)$, ..., $\underset{\sim}{x}(8)$.

Since the control sequence $u(k)$ is given by

$$u(k) = -\underset{\sim}{K}(k)\underset{\sim}{x}(k)$$

we find

$$u(0) = -\underset{\sim}{K}(0)\underset{\sim}{x}(0) = -\begin{bmatrix} -0.2114 & 0.7197 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = -1.0166$$

$$u(1) = -\underset{\sim}{K}(1)\underset{\sim}{x}(1) = -\begin{bmatrix} -0.2114 & 0.7197 \end{bmatrix} \begin{bmatrix} 0.9834 \\ -0.0166 \end{bmatrix} = 0.2198$$

Similarly, we obtain $u(2)$, $u(3)$, ..., $u(7)$.

Finally, the minimum value of J is obtained as follows:

$$J_{min} = \frac{1}{2} \underset{\sim}{x}^*(0)\underset{\sim}{P}(0)\underset{\sim}{x}(0) = \frac{1}{2} \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 1.2705 & -0.2114 \\ -0.2114 & 1.7197 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= 5.1348$$

Table B-8-1

k	$P(k)$	$K(k)$	$x(k)$	$u(k)$
0	$\begin{bmatrix} 1.2705 & -0.2114 \\ -0.2114 & 1.7197 \end{bmatrix}$	$\begin{bmatrix} -0.2114 & 0.7197 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	-1.0166
1	$\begin{bmatrix} 1.2705 & -0.2114 \\ -0.2114 & 1.7197 \end{bmatrix}$	$\begin{bmatrix} -0.2114 & 0.7197 \end{bmatrix}$	$\begin{bmatrix} 0.9834 \\ -0.0166 \end{bmatrix}$	0.2198
2	$\begin{bmatrix} 1.2705 & -0.2114 \\ -0.2114 & 1.7197 \end{bmatrix}$	$\begin{bmatrix} -0.2114 & 0.7197 \end{bmatrix}$	$\begin{bmatrix} 0.2033 \\ -0.2885 \end{bmatrix}$	0.2506
3	$\begin{bmatrix} 1.2704 & -0.2114 \\ -0.2114 & 1.7197 \end{bmatrix}$	$\begin{bmatrix} -0.2114 & 0.7197 \end{bmatrix}$	$\begin{bmatrix} -0.0379 \\ -0.1395 \end{bmatrix}$	0.0924
4	$\begin{bmatrix} 1.2703 & -0.2113 \\ -0.2113 & 1.7194 \end{bmatrix}$	$\begin{bmatrix} -0.2113 & 0.7194 \end{bmatrix}$	$\begin{bmatrix} -0.0471 \\ -0.0282 \end{bmatrix}$	0.0103
5	$\begin{bmatrix} 1.2697 & -0.2118 \\ -0.2118 & 1.7176 \end{bmatrix}$	$\begin{bmatrix} -0.2118 & 0.7176 \end{bmatrix}$	$\begin{bmatrix} -0.0179 \\ 0.0057 \end{bmatrix}$	-0.0079
6	$\begin{bmatrix} 1.2560 & -0.2143 \\ -0.2143 & 1.7143 \end{bmatrix}$	$\begin{bmatrix} -0.2143 & 0.7143 \end{bmatrix}$	$\begin{bmatrix} -0.0022 \\ 0.0068 \end{bmatrix}$	-0.0053
7	$\begin{bmatrix} 1.1667 & -0.1667 \\ -0.1667 & 1.6667 \end{bmatrix}$	$\begin{bmatrix} -0.1667 & 0.6667 \end{bmatrix}$	$\begin{bmatrix} 0.0015 \\ 0.0026 \end{bmatrix}$	-0.0015
8	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.0011 \\ 0.0004 \end{bmatrix}$	0

Discretized Quadratic Optimal Control Problem

Consider the continuous time control system

$$\dot{x} = Ax + Bu \qquad 8.37$$

Where $u(t) = u(kT),\ KT \le t < (k+1)T$

And the performance index to be minimized is

$$J = \frac{1}{2} x^*(t_f) S x(t_f) + \frac{1}{2}\int_0^{t_f} \left[x^*(t) Q x(t) + u^*(t) R u(t) \right] dt \qquad 8.38$$

The discretized system equation is

$$x((k+1)T) = G(T)x(kT) + H(T)u(kT) \qquad 8.39$$

And the discretized performance index when $t_f = NT$ will become as follows:

$$J = \frac{1}{2} x^*(NT) S x(NT) + \frac{1}{2}\sum_{k=0}^{N-1} \left[x^*(KT) Q_1 x(KT) + 2x^*(KT) M_1 u(KT) + u^*(KT) R_1 u(KT) \right]$$

Detailed steps are skipped.

VIII.3. Steady-state Quadratic Optimal Control

Consider a steady state quadratic optimal control of a regulator. The plant is given as follows:

$$x(k+1) = Gx(k) + Hu(k) \qquad 8.40$$

For $N = \infty$, the performance index may be modified to

$$J = \left\{ \frac{1}{2} \sum_{k=0}^{\infty} \left[x^*(k)Qx(k) + u^*(k)Ru(k) \right] \right\} \qquad 8.41$$

$\frac{1}{2} x^*(N)Qx(N)$, is not included in the J. This is because, if the optimal regular system is stable so that the value of J converges to a constant, $x(\infty) \to 0$ and $\frac{1}{2} x^*(N)Qx(N) \to 0$

Matrix P can be determined as follows:

$$P = Q + G^*P\left[I + HR^{-1}H^*P\right]^{-1}G = Q + G^*\left[P^{-1} + HR^{-1}H^*\right]^{-1}G \qquad 8.42$$

P can also be expressed in following

$$P = Q + G^*PG - G^*PH\left[R + H^*PH\right]^{-1}H^*PG \qquad 8.43$$

The steady state gain matrix K can be obtained in terms of P as follows

$$K = R^{-1}H^*(G^*)^{-1}(P - Q) \qquad 8.44$$

$$K = R^{-1}H^*\left(P^{-1} + HR^{-1}H^*\right)^{-1}G \qquad 8.45$$

$$K = \left(R + H^*PH\right)^{-1}H^*PG \qquad 8.46$$

$$u(k) = -Kx(k) \qquad 8.47$$

Substation of 8.47 into equation 8.46, we obtain:

$$u(k) = -\left(R + H^*PH\right)^{-1}H^*PGx(k) \qquad 8.48$$

$$x(k+1) = \left[G - H(R + H^* PH)^{-1} H^* PG\right] x(k)$$
$$= \left[I + H(R^{-1} H^* P)^{-1}\right] Gx(k) \qquad 8.49$$

The performance index J associated with the steady-state optimal control law can be obtained from equation 8.36 by substituting P for $P(0)$

$$J_{\min} = x^*(0)Px(0) \qquad 8.50$$

Steady-state Riccati Equation

To solve for steady-state Riccati equation

$$P = Q + G^* PG - G^* PH\left[R + H^* PH\right]^{-1} H^* PG \qquad 8.43$$

Is to start with the following bib steady Riccati equation:

$$P(k) = Q + G^* P(k+1)G - G^* P(k+1)H\left[R + H^* P(k+1)H\right]^{-1} H^* P(k+1)G \qquad 8.44$$

By reversing the direction of time, we can modify the equation to following

$$P(k+1) = Q + G^* P(k)G - G^* P(k)H\left[R + H^* P(k)H\right]^{-1} H^* P(k)G \qquad 8.45$$

Initialize $P(0) = 0$

Matlab Approach to the solution of steady-state quadratic optimal control problem

Example 8.2 Consider the system

$$x(k+1) = Gx(k) + Hu(k)$$
$$G = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, H = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

The performance index J is given by

$$J = \left\{ \frac{1}{2} \sum_{k=0}^{\infty} \left[x^*(k)Qx(k) + u^*(k)Ru(k)\right] \right\}$$

Where

$$Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, R = 1$$

The control law that minimizes J can be given by

$$u(k) = -Kx(k)$$

Determine the steady state gain matrix K.

```
%example8.2
% steady state quadratic optimal control
% optimal feedback gain matrix K
%**********enter matrices G,H,Q, and R***
clear all;
close all;

G=[1 0; 0 2]
H=[ 2; 2]
Q=[ 0.5 0; 0 0.5]
R=[1]


% start with the solution of steady state Riccati equation
% with P= [ 0 0; 0 0]
P= [ 0 0; 0 0]
P=Q+G'*P*G-G'*P*H*inv(R+H'*P*H)*H'*P*G;

% check solution every 20 steps of iteration

for i=1:20,
   P=Q+G'*P*G-G'*P*H*inv(R+H'*P*H)*H'*P*G;
end
P
for i=1:20,
   P=Q+G'*P*G-G'*P*H*inv(R+H'*P*H)*H'*P*G;
end
P

for i=1:20,
   P=Q+G'*P*G-G'*P*H*inv(R+H'*P*H)*H'*P*G;
end
P
%*** Optimalfeedback gain matrix K is obtained from

K=inv(R+H'*P*H)*H'*P*G
```

Lyapunov Approach to the solution of the Steady state Quadratic Optimal Regulator Problem

Let us consider the system

$$x(k+1) = Gx(k) \qquad 8.46$$

Where matrix G involves one or more adjustable parameters and all the eigenvalues of G lie inside the unit circle, or the origin x=0 is asymptotically stable.

Performance index

$J = \left\{ \frac{1}{2} \sum_{k=0}^{\infty} [x^*(k) Q x(k)] \right\}$, where Q is appositive definite or positive semidefinite Hermitian.

For 8.46, Lypunov function may be given by

$$V(x(k)) = x^*(k) P x(k)$$

Where P is a positive definite Hermitian matrix
And

$$\Delta V(x(k)) = \Delta V(x(k+1)) - \Delta V(x(k))$$
$$= x^*(k+1) P x(k+1) - x^*(k) P x(k)$$

Let us set

$$\begin{aligned} x^*(k) Q x(k) &= -[x^*(k+1) P x(k+1) - x^*(k) P x(k)] \\ &= -[(Gx(k))^* P (Gx(k)) - x^*(k) P x(k)] \\ &= -[x^*(k) G^* P G x(k) - x^*(k) P x(k)] \\ &= -x^*(k)[G^* P G - P] x(k) \end{aligned} \quad 8.47$$

By the second method of Lyapunov, we know that for a given matrix Q there exists a positive definite matrix P, since matrix G is stable, such that

$$-Q = G^* P G - P \quad 8.48$$

The performance index can be evaluates as follows:

$$J = \left\{ \frac{1}{2} \sum_{k=0}^{\infty} [x^*(k) Q x(k)] = \frac{1}{2} \sum_{k=0}^{\infty} [x^*(k) P x(k) - x^*(k+1) P x(k+1)] \right\}$$
$$= \frac{1}{2} x^*(0) P x(0)$$

Lyapunov Approach to the solution of the Steady state Quadratic Optimal Control Problem

$$x(k+1) = Gx(k) + Hu(k) \quad 8.49$$

Control Law:

$$u(k) = -Kx(k) \quad 8.50$$

Performance index:

$$J = \left\{ \frac{1}{2} \sum_{k=0}^{\infty} [x^*(k) Q x(k) + u^*(k) R u(k)] \right\} \quad 8.51$$

Substituting 8.50 into 8.49, we obtain

$$x(k+1) = Gx(k) - HKx(k) = (G - HK)x(k) \qquad 8.52$$

Substituting 8.50 into 8.51, we obtain

$$\begin{aligned} J &= \left\{\frac{1}{2}\sum_{k=0}^{\infty}\left[x^*(k)Qx(k) + (-Kx(k))^* R(-Kx(k))\right]\right\} \\ &= \left\{\frac{1}{2}\sum_{k=0}^{\infty}\left[x^*(k)Qx(k) + x^*(k)K^*RKx(k)\right]\right\} \\ &= \left\{\frac{1}{2}\sum_{k=0}^{\infty} x^*(k)[Q + K^*RK]x(k)\right\} \end{aligned} \qquad 8.53$$

We assume $(G - HK)$ is stable.

$$\begin{aligned} x^*(k)[Q + K^*RK]x(k) &= -[x^*(k+1)Px(k+1) - x^*(k)Px(k)] \\ &= -[((G - HK)x(k))^* P((G - HK)x(k)) - x^*(k)Px(k)] \\ &= -[x^*(k)(G - HK)^* P(G - HK)x(k) - x^*(k)Px(k)] \\ &= -x^*(k)[(G - HK)^* P(G - HK) - P]x(k) \end{aligned} \qquad 8.54$$

To have above equation true for any $x(k)$, we require

$$Q + K^*RK = -(G - HK)^* P(G - HK) + P \qquad 8.55$$

8.55 can be rewritten as

$$\begin{aligned} &Q + K^*RK + (G - HK)^* P(G - HK) - P = 0 \\ &\Rightarrow Q + G^*PG - P + K^*(R + H^*PH)K - (K^*H^*PG + G^*PHK) = 0 \end{aligned} \qquad 8.56$$

Matrix K that minimize J can be obtained as

$$K = (R + H^*PH)^{-1} H^*PG \qquad 8.57$$

Substituting 8.57 into 8.56 gives

$$P = Q + G^*P\left(I + HR^{-1}H^*P\right)^{-1}G$$

Finally minimum value of J can be obtained as follows:

$$J=\left\{\frac{1}{2}\sum_{k=0}^{\infty}x^*(k)[Q+K^*RK]x(k)\right\}$$
$$=\left\{\frac{1}{2}\sum_{k=0}^{\infty}x^*(k)Px(k)-x^*(k+1)Px(k+1)\right\}$$
$$=\frac{1}{2}x^*(0)Px(0)$$

Example 8.3

Consider the system defined by

$$\begin{bmatrix}\dot{x}_1(k+1)\\ \dot{x}_2(k+1)\end{bmatrix}=\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u(k),\ \text{and}\ \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

Performance index:

$$J=\left\{\frac{1}{2}\sum_{k=0}^{\infty}[x^*(k)Qx(k)+u^*(k)Ru(k)]\right\},$$

Where

$$Q=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ R=1$$

Determine the optimal control law to minimize the performance index.

we have

$$\mathbf{G}=\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix},\qquad \mathbf{H}=\begin{bmatrix}1\\ 0\end{bmatrix}$$

Matrix **P** can be determined from Equation (8–101), or

$$\mathbf{P}=\mathbf{Q}+\mathbf{G}^*\mathbf{P}(\mathbf{I}+\mathbf{H}R^{-1}\mathbf{H}^*\mathbf{P})^{-1}\mathbf{G}$$

Since matrices **Q**, **G**, **H**, and R are real, matrix **P** is a real symmetric matrix. By substituting given matrices **Q**, **G**, **H**, and R into Equation , we obtain

$$\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}[1\quad 0]\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\right)^{-1}\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$$

Simplifying this last equation, we get

$$\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+\frac{1}{1+p_{11}}\begin{bmatrix}p_{12} & p_{22}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}-p_{12} & -p_{12}\\ 1+p_{11} & 1+p_{11}\end{bmatrix}$$

or

$$\begin{bmatrix} p_{11}(1+p_{11}) & p_{12}(1+p_{11}) \\ p_{12}(1+p_{11}) & p_{22}(1+p_{11}) \end{bmatrix} = \begin{bmatrix} 1+p_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -p_{12}^2 + p_{22}(1+p_{11}) & -p_{12}^2 + p_{22}(1+p_{11}) \\ -p_{12}^2 + p_{22}(1+p_{11}) & -p_{12}^2 + p_{22}(1+p_{11}) \end{bmatrix}$$

This last equation is equivalent to the following three equations:

$$p_{11}(1+p_{11}) = 1 + p_{11} - p_{12}^2 + p_{22}(1+p_{11})$$
$$p_{12}(1+p_{11}) = -p_{12}^2 + p_{22}(1+p_{11})$$
$$p_{22}(1+p_{11}) = -p_{12}^2 + p_{22}(1+p_{11})$$

Solving these three equations for p_{11}, p_{12}, and p_{22}, requiring that $p_{11} > 0$, we obtain

$$p_{11} = 1, \qquad p_{12} = 0, \qquad p_{22} = 0$$

Hence

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{8–138}$$

Equation (8–138) gives the required solution of the steady-state Riccati equation.

Referring to Equation (8–79), we have

$$\begin{aligned} u(k) &= -(R + \mathbf{H}^*\mathbf{P}\mathbf{H})^{-1}\mathbf{H}^*\mathbf{P}\mathbf{G}\mathbf{x}(k) \\ &= -(1+1)^{-1}[1 \quad 0]\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\mathbf{x}(k) \\ &= -2^{-1}[0 \quad 0]\mathbf{x}(k) = 0 \end{aligned} \tag{8–139}$$

Equation (8–139) gives the optimal control law.

The closed-loop system now becomes

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\mathbf{x}(k) \tag{8–140}$$

Equation (8–140) gives the optimal closed-loop operation for the system. The closed-loop poles are at $\mu_1 = 1$ and $\mu_2 = 0$. The closed-loop system is not asymptotically stable.

The minimum value of J is obtained from Equation (8–81), as follows:

$$J_{\min} = \frac{1}{2}\mathbf{x}^*(0)\mathbf{P}\mathbf{x}(0) = \frac{1}{2}[1 \quad 1]\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}$$

Although the system is not asymptotically stable, the performance index becomes finite and is minimum. In fact, since $u(k) = 0$ for $k = 0, 1, 2, \ldots$, the system equation becomes

$$x_1(k+1) = 0$$
$$x_2(k+1) = x_1(k) + x_2(k)$$

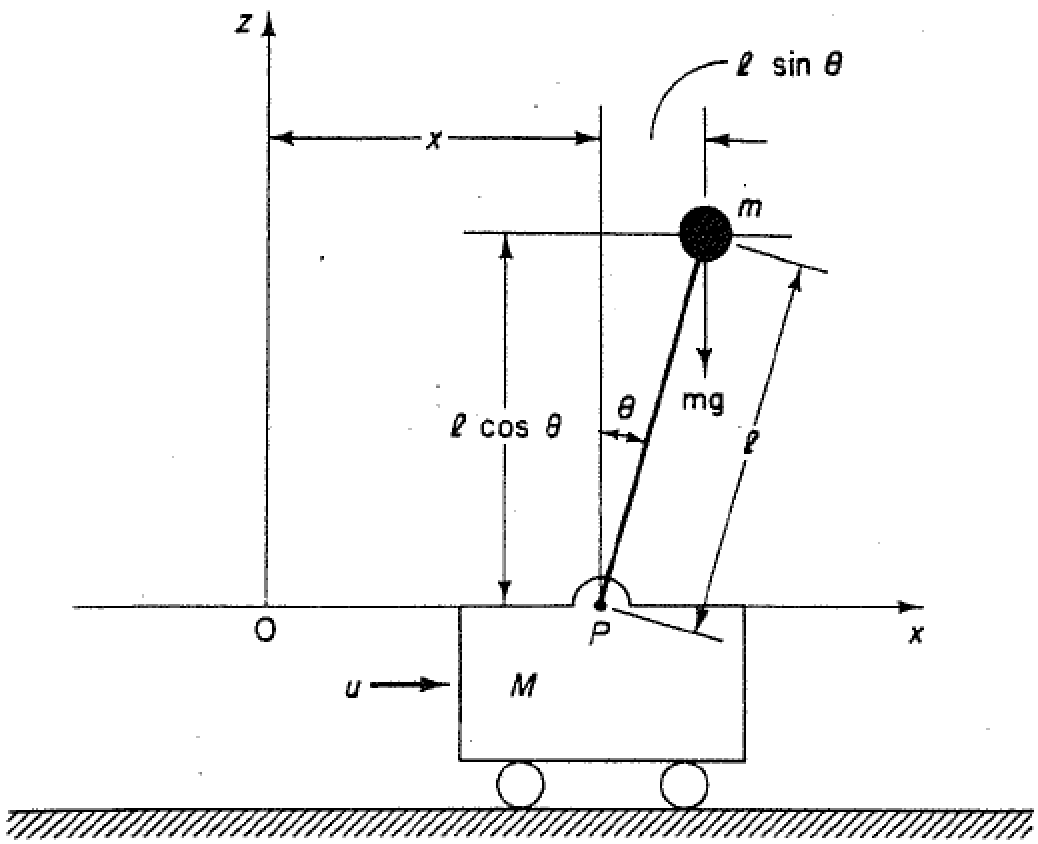
or

$$x_1(0) = 1, \qquad x_1(k) = 0, \qquad k = 1, 2, 3, \ldots$$
$$x_2(0) = 1, \qquad x_2(k) = 2, \qquad k = 1, 2, 3, \ldots$$

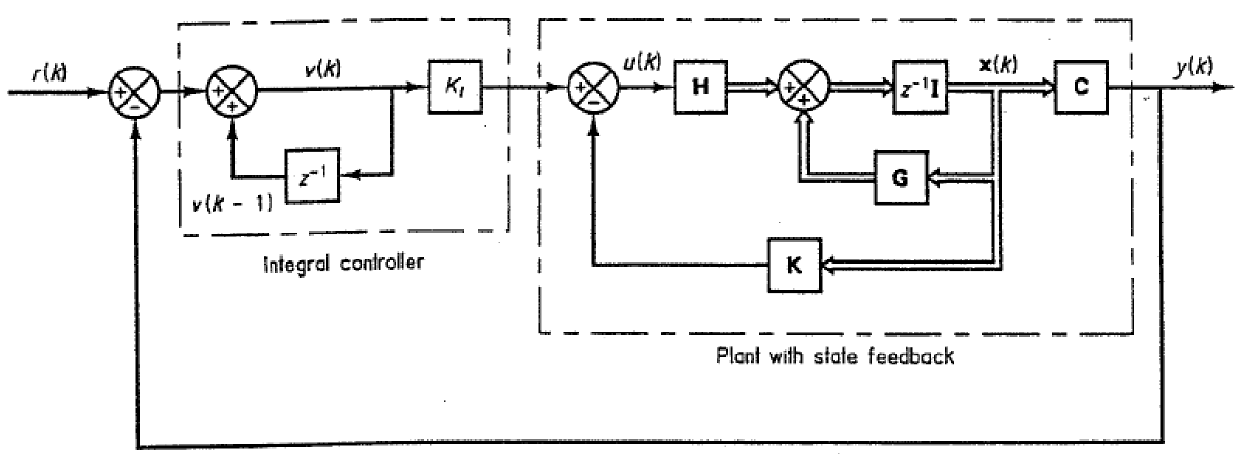
Notice that the performance index becomes finite, because it involves $x_1(k)$, but does not include $x_2(k)$. This example problem has shown that in an academic but not practical case, the quadratic optimal control does not yield an asymptotically stable system.

VIII.4. Quadratic Optimal Control of a Servo System

We consider a planar inverted pendulum. Assume the pendulum mass is concentrated at the end of the rod. The control force u is applied to the cart.



It is desired to keep the inverted pendulum upright as much as possible and yet control the position of the cart, for instance, by moving the cart in a step position. The inverted pendulum doesn't have a integrator, we need to build a type 1 servo system. We choose the sampling period T to be 0.1 sec.



Let us define state variable, x_1, x_2, x_3 and x_4

$$
\begin{aligned}
x_1 &= \theta \\
x_2 &= \dot{\theta} \\
x_3 &= x \\
x_4 &= \dot{x}
\end{aligned}
$$

We consider the displacement of the cart as output

$$
y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

We assume the following numerical value for M, m, and l
$M = 2kg$, $m = 0.1kg$, $l = 0.5m$
The continuous model:

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{aligned}
$$

Using the MATLAB command, we can discretize the model
$[G, H] = c2d(A, B, T)$

$$
\begin{aligned}
x(k+1) &= Gx(k) + Hu(k) \\
y(k) &= Cx(k) + Du(k)
\end{aligned}
$$

Thus the state-space representation for the system is following:

$$
\begin{aligned}
x(k+1) &= Gx(k) + Hu(k) \\
y(k) &= Cx(k) \\
v(k) &= v(k-1) + r(k) - y(k) \Rightarrow v(k+1) = v(k) + r(k+1) - y(k+1) \\
u(k) &= -Kx(k) + K_I v(k)
\end{aligned}
$$

where
$K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}$

Since

$$
\begin{aligned}
v(k+1) &= v(k) + r(k+1) - y(k+1) \Rightarrow \\
v(k+1) &= v(k) + r(k+1) - Cx(k+1) \\
&= v(k) + r(k+1) - C(Gx(k) + Hu(k)) \\
&= -CGx(k) + v(k) - CHu(k) + r(k+1)
\end{aligned}
$$

We have

$$\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} G & 0 \\ -CG & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} H \\ -CH \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(k+1)$$

Let us assume input r is a step function, or $r(k) = r(k+1) = r$

Define $x_e(k) = x(k) - x(\infty)$, and $v_e(k) = v(k) - v(\infty)$
Then the error equation becomes

$$\begin{bmatrix} x_e(k+1) \\ v_e(k+1) \end{bmatrix} = \begin{bmatrix} G & 0 \\ -CG & 1 \end{bmatrix} \begin{bmatrix} x_e(k) \\ v_e(k) \end{bmatrix} - \begin{bmatrix} H \\ -CH \end{bmatrix} u_e(k)$$

Define $\hat{G} = \begin{bmatrix} G & 0 \\ -CG & 1 \end{bmatrix}$, $\hat{H} = \begin{bmatrix} H \\ -CH \end{bmatrix}$, $\hat{K} = \begin{bmatrix} K & -K_I \end{bmatrix}$, $w(k) = u_e(k)$

$$\xi(k) = \begin{bmatrix} x_e(k) \\ v_e(k) \end{bmatrix} = \begin{bmatrix} x_{1e}(k) \\ x_{2e}(k) \\ x_{3e}(k) \\ x_{4e}(k) \\ x_{5e}(k) \end{bmatrix}, \text{ where } x_{5e}(k) = v_e(k)$$

Then we have

$$\xi(k+1) = \hat{G}x(k) + \hat{H}w(k)$$

$$w(k) = -\hat{K}\xi(k)$$

Performance index is as follows:

$$J = \left\{ \frac{1}{2} \sum_{k=0}^{\infty} \left[\xi^*(k) Q \xi(k) + w^*(k) R w(k) \right] \right\}$$

Thus we can find P and K

$$P = Q + G^* P \left(I + H R^{-1} H^* P \right)^{-1} G$$
$$K = \left(R + H^* P H \right)^{-1} H^* P G$$

Step Response:

Once K is determined, we can determine the unit step response

$$\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} G & 0 \\ -CG & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} H \\ -CH \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(k+1)$$

$$= \begin{bmatrix} G & 0 \\ -CG & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} - \begin{bmatrix} H \\ -CH \end{bmatrix} \begin{bmatrix} K & -K_I \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(k+1)$$

$$= \begin{bmatrix} G-HK & HK_I \\ -CG+CHK & 1-CHK_I \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} c & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + [0] r$$

Let

$$GG = \begin{bmatrix} G-HK & HK_I \\ -CG+CHK & 1-CHK_I \end{bmatrix}, \quad HH = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad CC = \begin{bmatrix} C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \; D = [0]$$

Using MatLAB command to find the unit step response

$[num, den] = ss2tf(GG, HH, CC, DD)$, $y = filter(num, den, r)$

Example 8.3 & 8.4