VI Pole Placement and Observer Design

VI.1. Introduction

Controllability and observability play an important role in the optimal control of multivariable systems.

Pole placement will be used to design the controller and the observer.

VI.2. Controllability

A control system is said to be completely state controllable if it is possible to transfer the system from any arbitrary initial state to any desired state in a finite time period.

Consider the discrete time control system defined by

$$x((k+1)T) = Gx(kT) + Hu(kT) \qquad 6.1$$

Where

$x(kT)$=n-vector	(state vector at kth sampling instant)
$u(kT)$	(control signal at kth sampling instant)
$G = n \times n$ matrix	
$H = n \times 1$ matrix	
T = Sampling Period	

We assume that $u(kT)$ is constant for $kT \le t \le (k+1)T$

Define controllability matrix $\left[H \;\vdots\; GH \;\vdots\; \cdots \;\vdots\; G^{n-1}H\right]$

The condition for complete state controllability is that the $n \times n$ matrix $\left[H \;\vdots\; GH \;\vdots\; \cdots \;\vdots\; G^{n-1}H\right]$ be of rank n, or that

Rank $\left[H \;\vdots\; GH \;\vdots\; \cdots \;\vdots\; G^{n-1}H\right] = n$

It is possible to find n linearly independent scalar equations from which a sequence of unbounded control signals $u(kT) \quad (k = 0,1,2,\cdots n-1)$ can be uniquely determined such that any initial state $x(0)$ is transferred to the desired state in n sampling periods.

Solutions of the system 6.1

$$x(nT) - G^n x(0) = \left[H \;\vdots\; GH \;\vdots\; \cdots \;\vdots\; G^{n-1}H\right] \begin{bmatrix} u((n-1)T) \\ u((n-2)T) \\ \vdots \\ u(0) \end{bmatrix}$$

Alternative form of the condition for complete state controllability

Consider the discrete time control system defined by

$$x((k+1)T) = Gx(kT) + Hu(kT) \qquad 6.1$$

Where

$x(kT)$=n-vector	(state vector at kth sampling instant)
$u(kT)$=r-vector	(control signal at kth sampling instant)

$G = n \times n$ matrix
$H = n \times r$ matrix
T = Sampling Period

If the eigenvectors of G are distinct, then it is possible to find a transformation matrix P such that

$$P^{-1}GP = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Note: if the eigenvalues of G are distinct then the eigenvectors of G are distinct. The converse is not true.

The condition for complete state controllability is that, if the eigenvectors of G are distinct, then the system is completely state controllable if and only if no row of $P^{-1}H$ has all zero elements.

If the G matrix does not possess distinct eigenvectors, then diagnalization is impossible. We may transform G into a Jordan canonical form.

Example for Jordan canonical form:

$$J = \begin{bmatrix} \lambda 1 & 1 & 0 & & & & & & \\ & \lambda 1 & 1 & & & & & & \\ & & \lambda 1 & & & & & & \\ & & & \lambda 2 & 1 & & & & \\ & & & & \lambda 2 & & & & \\ & & & & & \lambda 3 & & & 0 \\ & & & & & & \lambda 4 & & \\ & & & & & & & \lambda 5 & \\ & & & & & 0 & & & \lambda 6 \end{bmatrix}$$

Assume we can find a transformation matrix S such that

$$S^{-1}GS = J$$

If we define a new state vector $\hat{x}$ by $x(kT) = S\hat{x}(kT)$, substitute into 6.1,

$$\hat{x}((k+1)T) = S^{-1}GS\hat{x}(kT) + S^{-1}Hu(kT)$$
$$= J\hat{x}(kT) + S^{-1}Hu(kT) \qquad 6.2$$

The condition for the complete state controllability of the system of above equation is as follows:

The system is completely state controllable if and only if 1) no two Jordan blocks in J of 6.2 are associated with the same eigenvalues, 2) The elements of any row of $S^{-1}H$ that corresponds to the last row of each Jordan block are not all zero and 3) The elements of each row of $S^{-1}H$ that correspond to distinct eigenvalues are not all zero.

Example 6.1 Consider the system defined by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Determine the conditions on a,b,c and d for complete state controllability.

$$rank\left[H \;\; \vdots \;\; GH\right] = rank\begin{bmatrix} 1 & a+b \\ 1 & c+d \end{bmatrix} = 2 \rightarrow a+b \neq c+d$$

Complete Output Controllability

$$x((k+1)T) = Gx(kT) + Hu(kT) \qquad 6.3$$
$$y(kT) = Cx(kT) \qquad 6.4$$

Where

$x(kT)$=n-vector (state vector at kth sampling instant)
$u(kT)$ (control signal at kth sampling instant)
$y(kT)$=m-vector (output vector at kth sampling instant)
$G = n \times n$ matrix
$H = n \times 1$ matrix
$C = m \times n$ matrix

T = Sampling Period

Define output controllability matrix $\left[CH \;\; \vdots \;\; CGH \;\; \vdots \;\; \cdots \;\; \vdots \;\; CG^{n-1}H\right]$
The condition for complete output controllability is that the matrix

$\left[C H \;\vdots\; CGH \;\vdots\; \cdots \;\vdots\; CG^{n-1}H\right]$ be of rank m, or that
Rank $\left[C H \;\vdots\; CGH \;\vdots\; \cdots \;\vdots\; CG^{n-1}H\right] = m$

The system defined by equation 6.3 and 6.4 is said to be completed output controllable, or simply output controllable, if it is possible to construct an unconstrained control signal $u(kT)$ defined over a finite number of sampling periods $0 \le kT \le nT$ such that, starting from any initial output $y(0)$, the output $y(kT)$ can be transferred to the desired point $y(f)$ in the output space in at most n sampling periods.

$$x(nT) - G^n x(0) = \left[H \;\vdots\; GH \;\vdots\; \cdots \;\vdots\; G^{n-1}H\right] \begin{bmatrix} u((n-1)T) \\ u((n-2)T) \\ \vdots \\ u(0) \end{bmatrix}$$

Thus

$$Cx(nT) - CG^n x(0) = \left[C H \;\vdots\; CGH \;\vdots\; \cdots \;\vdots\; CG^{n-1}H\right] \begin{bmatrix} u((n-1)T) \\ u((n-2)T) \\ \vdots \\ u(0) \end{bmatrix}$$

So:

$$y(nT) - CG^n x(0) = \left[C H \;\vdots\; CGH \;\vdots\; \cdots \;\vdots\; CG^{n-1}H\right] \begin{bmatrix} u((n-1)T) \\ u((n-2)T) \\ \vdots \\ u(0) \end{bmatrix}$$

Next, consider the system defined by the equations:

$$x((k+1)T) = Gx(kT) + Hu(kT) \qquad 6.5$$
$$y(kT) = Cx(kT) + Du(kT) \qquad 6.6$$

Where

$x(kT)$=n-vector (state vector at kth sampling instant)
$u(kT)$=r-vector (control signal at kth sampling instant)
$y(kT)$=m-vector (output vector at kth sampling instant)

$G = n \times n$ matrix
$H = n \times r$ matrix
$C = m \times n$ matrix
$D = m \times r$ matrix

T = Sampling Period

Define output controllability matrix $\left[D \;\vdots\; CH \;\vdots\; CGH \;\vdots\; \cdots \;\vdots\; CG^{n-1}H\right]$
The condition for complete output controllability is that the matrix
$\left[D \;\vdots\; CH \;\vdots\; CGH \;\vdots\; \cdots \;\vdots\; CG^{n-1}H\right]$ be of rank m, or that
Rank $\left[D \;\vdots\; CH \;\vdots\; CGH \;\vdots\; \cdots \;\vdots\; CG^{n-1}H\right] = m$

$$Cx(nT) - CG^{n}x(0) = \left[CH \;\vdots\; CGH \;\vdots\; \cdots \;\vdots\; CG^{n-1}H\right]\begin{bmatrix} u((n-1)T) \\ u((n-2)T) \\ \vdots \\ u(0) \end{bmatrix}$$

$$y(nT) = Cx(nT) + Du(nT)$$

$$= CG^{n}x(0) + \left[CH \;\vdots\; CGH \;\vdots\; \cdots \;\vdots\; CG^{n-1}H\right]\begin{bmatrix} u(((n-1)T))nT \\ u((n-2)T) \\ \vdots \\ u(0) \end{bmatrix} + Du(nT)$$

$$= CG^{n}x(0) + \left[D \;\vdots\; CH \;\vdots\; CGH \;\vdots\; \cdots \;\vdots\; CG^{n-1}H\right]\begin{bmatrix} u(nT) \\ u((n-1)T) \\ u((n-2)T) \\ \vdots \\ u(0) \end{bmatrix}$$

Controllability of a Linear Time-Invariant continuous time control system.

Consider the system defined by

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

x = state vector (n-vector)
u = control vector (r-vector)
y = output vector (m- vector)

A = $n \times n$ matrix
B = $n \times r$ matrix
C = $m \times n$ matrix
D = $m \times r$ matrix

Complete state controllability.

$$rank\left[B \;\vdots\; AB \;\vdots\; \cdots \;\vdots\; A^{n-1}B\right] = n$$

Output controllability

$$rank\left[D \;\vdots\; CB \;\vdots\; CAB \;\vdots\; \cdots \;\vdots\; CA^{n-1}B\right] = m$$

VI.3. Observability

Consider the unforced discrete time control system defined by

$$x((k+1)T) = Gx(kT) \qquad 6.7$$
$$y(kT) = Cx(kT) \qquad 6.8$$

Where

$x(kT)$=n-vector (state vector at kth sampling instant)
$y(kT)$=m-vector (output vector at kth sampling instant)
$G = n \times n$ matrix
$C = m \times n$ matrix

T = Sampling Period

The system is said to be completely observable if every initial state $x(0)$ can be determined from the observation $y(kT)$ over a finite number of sampling periods.

Or

$$rank\left[C^* \; \vdots \; G^*C^* \; \vdots \; \cdots \; \vdots \; (G^*)^{n-1}C^*\right] = n$$

Alternative form of the condition for complete observability

$$x((k+1)T) = Gx(kT) \qquad 6.9$$
$$y(kT) = Cx(kT) \qquad 6.10$$

Suppose the eigenvalues of G are distinct, and a transformation matrix P transforms G into a diagonal matrix, so that $P^{-1}GP$ is a diagonal matrix.

Define :

$$x(kT) = P\hat{x}(kT)$$

Then 6.9 and 6.10 can be written as follows:

$$\hat{x}((k+1)T) = P^{-1}GP\hat{x}(kT)$$
$$y(kT) = Cx(kT) = CP\hat{x}(kT)$$
$$\Rightarrow y(nT) = CP(P^{-1}GP)^n\hat{x}(0)$$

$$\left(P^{-1}GP\right)^n = \begin{bmatrix} \lambda_1^n & & & 0 \\ & \lambda_2^n & & \\ & & \ddots & \\ 0 & & & \lambda_n^n \end{bmatrix}$$

$$y(nT) = CP\left(P^{-1}GP\right)^n \hat{x}(0) = CP\begin{bmatrix} \lambda_1^n & & & 0 \\ & \lambda_2^n & & \\ & & \ddots & \\ 0 & & & \lambda_n^n \end{bmatrix}\hat{x}(0) = CP\begin{bmatrix} \lambda_1^n \hat{x}_1(0) \\ \lambda_2^n \hat{x}_2(0) \\ \vdots \\ \lambda_n^n \hat{x}_n(0) \end{bmatrix}$$

The system is completely observable if and only if none of the columns of the $m \times n$ matrix CP consists of all zero elements.

Note: if the ith column of CP consists of all zero elements, then the state variable $\hat{x}_i(0)$ will not appear in the output equation and therefore can not be determined from observation of $y(kT)$. Thus x(0), which is related to $\hat{x}(0)$ by P, can not be determined.

If the matrix G involves multiple engenvalues, then G may be transformed into Jordan canonical form:

$$S^{-1}GS = J$$

If we define a new state vector $\hat{x}$ by $x(kT) = S\hat{x}(kT)$, substitute into 6.7,

$$\begin{aligned} \hat{x}((k+1)T) &= S^{-1}GS\hat{x}(kT) \\ &= J\hat{x}(kT) \end{aligned}$$

$$y(kT) = CS\hat{x}(kT) \Rightarrow y(nT) = CS\left(S^{-1}GS\right)^n \hat{x}(0)$$

The condition for the complete observalibility of the system of above equation is as follows:

The system is completely observable if and only if 1) no two Jordan blocks in J are associated with the same eigenvalues, 2) none of the columns of CS that corresponds to the first row of each Jordan block consists of all zero elements 3) The elements of each column of CS that correspond to distinct eigenvalues are not all zero.

Example 6.2 Consider the system defined by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Determine the conditions on a, b, c and d for complete observability.

$$rank[C^* \;\vdots\; G^* C^*] = rank \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} = 2 \rightarrow b \neq 0$$

Condition for complete observability in the z plane:

Note: a necessary and sufficient condition for complete observability is that no pole-zero cancellation occur in the pulse transfer function. If cancellation occurs, the canceled mode can not be observed in the output.

Principle of Duality.

Consider the system S1 defined by the equations

$$x((k+1)T) = Gx(kT) + Hu(kT) \qquad 6.11$$
$$y(kT) = Cx(kT) \qquad 6.12$$

Where

$x(kT)$=n-vector (state vector at kth sampling instant)
$u(kT)$=r-vector (control signal at kth sampling instant)
$y(kT)$=m-vector (output vector at kth sampling instant)
$G = n \times n$ matrix
$H = n \times r$ matrix
$C = m \times n$ matrix

T = Sampling Period

Consider S1 counterpart, S2 defined by the equations:

$$\hat{x}((k+1)T) = G^* \hat{x}(kT) + C^* \hat{u}(kT) \qquad 6.13$$
$$\hat{y}(kT) = H^* \hat{x}(kT) \qquad 6.14$$

Where

$\hat{x}(kT)$=n-vector (state vector at kth sampling instant)
$\hat{u}(kT)$=m-vector (control signal at kth sampling instant)
$\hat{y}(kT)$=r-vector (output vector at kth sampling instant)
G^*= conjugate transpose of G

$H*=$ conjugate transpose of H
$C*=$ conjugate transpose of C

T = Sampling Period

Note: The analogy between controllability and observability is referred to as the principle of duality, due to kalman. The principle of duality states that the system S1 defined by equations 6.11-12 is completely state controllable (observable) if and only if system S2 defined by equation 6.13-14 is completely observable (state Controllable).

For system S1:

1) A necessary and sufficient condition for complete state controllability is that

$$rank\left[H \;\vdots\; GH \;\vdots\; \cdots \;\vdots\; G^{n-1}H\right] = n$$

2) A necessary and sufficient condition for complete observability is that

$$rank\left[C* \;\vdots\; G*C* \;\vdots\; \cdots \;\vdots\; (G*)^{n-1}C*\right] = n$$

For system S2:

1) A necessary and sufficient condition for complete state controllability is that

$$rank\left[C* \;\vdots\; G*C* \;\vdots\; \cdots \;\vdots\; (G*)^{n-1}C*\right] = n$$

2) A necessary and sufficient condition for complete observability is that

$$rank\left[H \;\vdots\; GH \;\vdots\; \cdots \;\vdots\; G^{n-1}H\right] = n$$

Complete Observability Linear Time-Invariant Continuous-Time Control system.

Consider the system defined by

$$\dot{x} = Ax$$

$$y = Cx$$

x = state vector (n-vector)
y = output vector (m- vector)

A = $n \times n$ matrix
C = $m \times n$ matrix

Complete observability is that the rank of the . $n \times nm$ matrix

$$rank\left[C^* \;\vdots\; A^*C^* \;\vdots\; \cdots \;\vdots\; (A^*)^{n-1}C^*\right] = n$$

Effects of the Discretization of a continuous-time control system on controllability and observability.

Note: It can be shown that a system that is completely state controllable and completely observable in the absence of sampling remains completely state controllable and completely observable after introduction of sampling if and only if, for every eigenvalue of the characteristic equation for the continuous-time control system, the relationship

$$\operatorname{Re}\lambda_i = \operatorname{Re}\lambda_j$$

Implies

$$\operatorname{Im}(\lambda_i - \lambda_j) \neq \frac{2n\pi}{T}$$

Where T is the sampling period and $n = \pm 1, \pm 2, \cdots$ It is noted that, unless the system contains complex poles, pole-zero cancellation will not occur in passing from the continuous-time to the discrete time case.

Example 6.3 Consider the continuous time control system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Determine the controllability and observability of the continuous time system and the corresponding discrete time control system.

$$rank\left[B \;\vdots\; AB\right] = rank\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = 2 \rightarrow controllable$$

$$rank\left[C^* \;\vdots\; A^*C^*\right] = rank\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = 2 \rightarrow observable$$

The eigenvalues of the state matrix are $\lambda_{i,2} = \pm 2j$

Discretizing the continuous time control system will be

$$G(T) == e^{AT} = \begin{bmatrix} \cos 2T & \sin 2T \\ -\sin 2T & \cos 2T \end{bmatrix},$$

$$H(T) == (e^{AT} - I)A^{-1}B = \begin{bmatrix} \cos 2T - 1 & \sin 2T \\ -\sin 2T & \cos 2T - 1 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5\cos 2T + 0.5 \\ 0.5\sin 2T \end{bmatrix}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \cos 2T & \sin 2T \\ -\sin 2T & \cos 2T \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} -0.5\cos 2T + 0.5 \\ 0.5\sin 2T \end{bmatrix} u(k)$$

$$y(kT) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix}$$

$$\operatorname{Im}(\lambda_i - \lambda_j) = 4 \neq \frac{2n\pi}{T}$$

Check:

$$rank\begin{bmatrix} H & \vdots & GH \end{bmatrix} = rank\begin{bmatrix} -0.5\cos 2T + 0.5 & -0.5\cos^2 2T + 0.5\cos 2T + 0.5\sin^2 2T \\ 0.5\sin 2T & 0.5\sin 2T \end{bmatrix}$$

Above matrix rank is 2 only if $0.5\sin 2T \neq 0 \Rightarrow T \neq \frac{n\pi}{2}$

VI.4. Useful Transformation

Transforming state-space equations into canonical forms

$$x(k+1) = Gx(k) + Hu(k) \qquad 6.15$$

$$y(k) = Cx(k) + Du(k) \qquad 6.16$$

We will transform the equations 6.15-6.16 into the following three canonical forms: 1) Controllable canonical form 2) observable canonical form, 3) diagonal or Jordan canonical form.

A) controllable canonical form

$T = MW$, $M = \left[H \;\vdots\; GH \;\vdots\; \cdots \;\vdots\; G^{n-1}H \right]$

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

The elements a_i shown in matrix W are coefficients of the characteristic equation

$$\left| zI - G \right| = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

Now let us define $x(k) = T\hat{x}(k)$

Then eq. 15-16 become

$$\hat{x}(k+1) = T^{-1}GT\hat{x}(k) + T^{-1}Hu(k) = \hat{G}\hat{x}(k) + \hat{H}u(k)$$

$$y(k) = CT\hat{x}(k) + Du(k) = \hat{C}\hat{x}(k) + \hat{D}u(k)$$

Where, $\hat{G} = T^{-1}GT$ *and* $\hat{H} = T^{-1}H$, $\hat{C} = CT$, $\hat{D} = D$, or

$$\begin{bmatrix} \hat{x}_1(k+1) \\ \hat{x}_2(k+1) \\ \vdots \\ \hat{x}_{n-1}(k+1) \\ \hat{x}_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \vdots \\ \hat{x}_{n-1}(k) \\ \hat{x}_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_n - a_n b_0 & \vdots & b_{n-1} - a_{n-1} b_0 & \vdots & \cdots & \vdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \vdots \\ \hat{x}_n(k) \end{bmatrix} + \hat{D}u(k)$$

Where b_k are those coefficients appearing in the numerator of the following pulse transfer function.

$$C(ZI-G)^{-1}H + D = \hat{C}(ZI-\hat{G})^{-1}\hat{H} + \hat{D} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots b_{n-1} z + b_n}{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}$$

B) Observable canonical form

$$Q = (WN^*)^{-1}, \; N = \begin{bmatrix} C^* & \vdots & G^*C^* & \vdots & \cdots & \vdots & (G^*)^{n-1}C^* \end{bmatrix}$$

$$Q^{-1}GQ = \hat{G} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & & 0 & -a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}$$

$$Q^{-1}H = \hat{H} = \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix}$$

And

$$CQ = \hat{C} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

By defining $x(k) = Q\hat{x}(k)$

Then eq. 15-16 become

$$\hat{x}(k+1) = \hat{G}\hat{x}(k) + \hat{H}u(k)$$
$$y(k) = \hat{C}\hat{x}(k) + \hat{D}u(k)$$

Where, $\widehat{G} = Q^{-1}GQ$ *and* $\widehat{H} = Q^{-1}H$, $\widehat{C} = CQ$, $\widehat{D} = D$, or

$$\begin{bmatrix} \hat{x}_1(k+1) \\ \hat{x}_2(k+1) \\ \vdots \\ \hat{x}_{n-1}(k+1) \\ \hat{x}_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & & 0 & -a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \vdots \\ \hat{x}_{n-1}(k) \\ \hat{x}_n(k) \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \vdots \\ \hat{x}_n(k) \end{bmatrix} + \widehat{D}u(k)$$

C) Diagonal Jordan Canonical Form

If the eigenvalues of the matrix G are distinct, the corresponding eigenvectors $[\xi_1 \quad \xi_2 \quad \cdots \quad \xi_n]$ are distinct.
Define the transformation matrix $P = [\xi_1 \quad \xi_2 \quad \cdots \quad \xi_n]$

$$P^{-1}GP = \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & P_n \end{bmatrix}$$

By defining $x(k) = P\hat{x}(k)$
Then eq. 15-16 become

$$\hat{x}(k+1) = \widehat{G}\hat{x}(k) + \widehat{H}u(k)$$
$$y(k) = \widehat{C}\hat{x}(k) + \widehat{D}u(k)$$

Where, $\widehat{G} = P^{-1}GP$ *and* $\widehat{H} = P^{-1}H$, $\widehat{C} = CP$, $\widehat{D} = D$, or

$$\begin{bmatrix} \hat{x}_1(k+1) \\ \hat{x}_2(k+1) \\ \vdots \\ \hat{x}_{n-1}(k+1) \\ \hat{x}_n(k+1) \end{bmatrix} = \begin{bmatrix} P_1 & 0 & \cdots & 0 & 0 \\ 0 & P_2 & \cdots & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & P_n \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \vdots \\ \hat{x}_{n-1}(k) \\ \hat{x}_n(k) \end{bmatrix} + \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{bmatrix} u(k)$$

$$y(k) = [\beta_1 \quad \beta_2 \quad \cdots \quad \beta_n] \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \vdots \\ \hat{x}_n(k) \end{bmatrix} + \hat{D}u(k)$$

Where the α_i and β_i are constants such that $\alpha_i \ \beta_i$ will appear in the numerator of the term $\frac{1}{z - p_i}$ when the pulse transfer function is expanded into partial fractions as follows:

$$C(ZI - G)^{-1} H + D = \hat{C}(ZI - \hat{G})^{-1} \hat{H} + \hat{D} = \frac{\alpha_1\beta_1}{z - p_1} + \frac{\alpha_2\beta_2}{z - p_2} + \cdots + \frac{\alpha_n\beta_n}{z - p_n} + D$$

In many cases we choose $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$.

Remarks: The sufficient and necessary conditions for the system to be completely state controllable is that $\alpha_i \neq 0$ and sufficient and necessary conditions for the system to be completely observable is $\beta_i \neq 0$.

The case for multiple eigenvalues p_i of matrix G then Jordan canonical form will be formed. (skipped)

Invariance property of the rank condition for the controllability matrix and observability matrix.

1) For controllability matrix $M = \left[H \ \vdots \ GH \ \vdots \ \cdots \ \vdots \ G^{n-1}H\right]$

Let P be a transformation matrix and $P^{-1}GP = \tilde{G}$, $P^{-1}H = \tilde{H}$

Then:
$$\begin{aligned} &P^{-1}G^2P = P^{-1}GPP^{-1}GP = \tilde{G}^2 \\ &P^{-1}G^3P = P^{-1}GPP^{-1}GPP^{-1}GP = \tilde{G}^3 \\ &\vdots \\ &P^{-1}G^{n-1}P = P^{-1}GPP^{-1}GPP^{-1}GP = \tilde{G}^{n-1} \end{aligned}$$

$$\begin{aligned}
P^{-1}M &= P^{-1}\left[H \;\vdots\; GH \;\vdots\; \cdots \;\vdots\; G^{n-1}H\right] \\
&= \left[P^{-1}H \;\vdots\; P^{-1}GH \;\vdots\; \cdots \;\vdots\; P^{-1}G^{n-1}H\right] \\
&= \left[P^{-1}H \;\vdots\; P^{-1}GPP^{-1}H \;\vdots\; \cdots \;\vdots\; P^{-1}G^{n-1}PP^{-1}H\right] \\
&= \left[\tilde{H} \;\vdots\; \tilde{G}\tilde{H} \;\vdots\; \cdots \;\vdots\; \tilde{G}^{n-1}\tilde{H}\right] \\
&= \tilde{M}
\end{aligned}$$

Since matrix P is no singular, rank M= rank $\tilde{M}$
Similarly, for the observability matrix
$N = \left[C* \;\vdots\; G*C* \;\vdots\; \cdots \;\vdots\; (G*)^{n-1}C*\right]$
Let P be a transformation matrix and $P^{-1}GP = \tilde{G}$, $CP = \tilde{C}$

Then

$$\begin{aligned}
P*N &= \left[P*C* \;\vdots\; P*G*C* \;\vdots\; \cdots \;\vdots\; P*(G*)^{n-1}C*\right] \\
&= \left[\tilde{C}* \;\vdots\; \tilde{G}*\tilde{C}* \;\vdots\; \cdots \;\vdots\; (\tilde{G}*)^{n-1}\tilde{C}*\right] \\
&= \tilde{N}
\end{aligned}$$

Since P is non singular, rank N= rank $\tilde{N}$

VI.5. Design Via Pole Placement

Remarks: if the system is completely state controllable, then poles of the closed-loop system may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix.

Necessary and sufficient conditions for arbitrary pole placement

The state equation is $x(k+1) = Gx(k) + Hu(k)$ 6-17

Where

$x(k)$ state vector at kth sampling instant
$u(k)$ control signal at kth sampling instant
G $n \times n$ matrix
H $n \times 1$ matrix

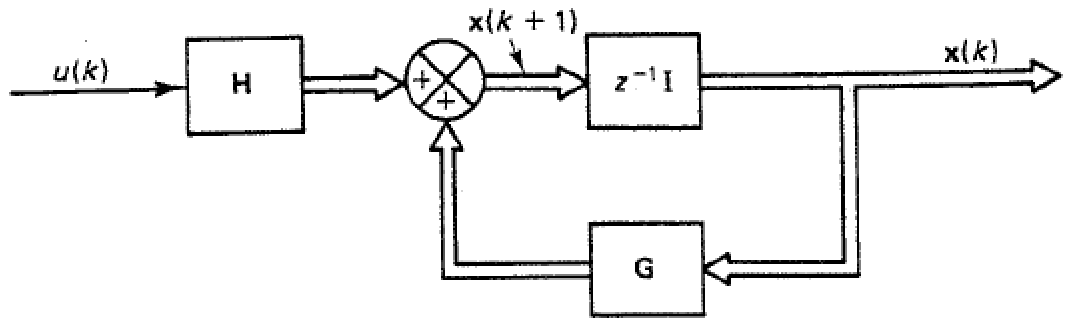


Figure 6.1

If the control signal is chosen as $u(k) = -Kx(k)$
Where K is the feedback gain matrix.

Then the system becomes a closed loop control system and its state equation becomes

$x(k+1) = (G - HK)x(k)$ 6-18

We will choose K such that the eigenvalues of $(G - HK)$ are desired closed-loop poles, $u_1, u_2, \cdots u_n$

Remarks: We can approve that a necessary and sufficient condition for arbitrary pole placement is that the system be completely state controllable.

1) For necessary conditions, we can assume that 6-17 is not completely state controllable. Then the controllability matrix is not full rank, then we can find out that matrix K can not control all the eigenvalues.

2) The sufficient conditions. We will approve that if the system is completely state controllable, then we will find matrix K that make the eigenvalues of $(G-HK)$ as desired.

The desired eigenvalues of $(G-HK)$ are $u_1, u_2, \cdots u_n$.
Noting the characteristic equation of the original system 6-17 is

$$|(ZI-G)| = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

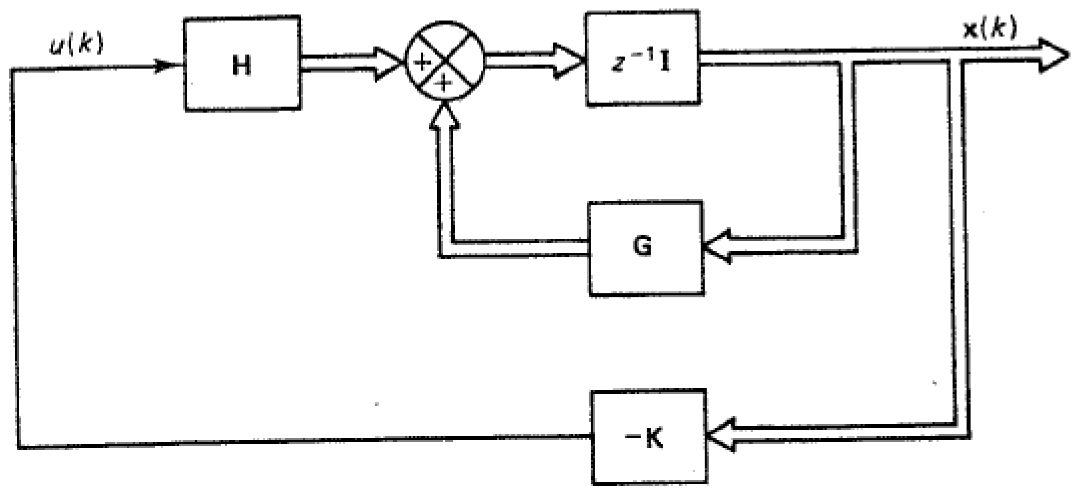


Figure 6.2

We define a transformation matrix T as follows:
$T = MW$, where $M = [H \ \vdots \ GH \ \vdots \ \cdots \ \vdots \ G^{n-1}H]$, which is of rank n,
And

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$a_1, a_2 \cdots a_{n-1}, a_n$ are from $|(ZI-G)| = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$
Then we will have

$$T^{-1}GT = \widehat{G} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

And

$$T^{-1}H = \widehat{H} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Next we define $\widehat{K} = KT = [\delta_n \quad \delta_{n-1} \quad \cdots \quad \delta_1]$

Then $\widehat{H}\widehat{K} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [\delta_n \quad \delta_{n-1} \quad \cdots \quad \delta_1] = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \delta_n & \delta_{n-1} & \delta_{n-2} & \cdots & \delta_1 \end{bmatrix}$

The characteristic equation $|(ZI - G + HK)|$

Becomes as follows:

$$|(ZI - G + HK)| = |(ZI - \widehat{G} + \widehat{H}\widehat{K})|$$

$$= \left| z\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \delta_n & \delta_{n-1} & \delta_{n-2} & \cdots & \delta_1 \end{bmatrix} \right|$$

$$= \left\| \begin{bmatrix} z & -1 & 0 & \cdots & 0 \\ 0 & z & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ a_n + \delta_n & a_{n-1} + \delta_{n-1} & a_{n-2} + \delta_{n-2} & \cdots & z + a_1 + \delta_1 \end{bmatrix} \right\|$$

$$= z^n + (a_1 + \delta_1)z^{n-1} + \cdots + (a_{n-1} + \delta_{n-1})z + a_n + \delta_n = 0$$

The characteristic equation with the desired eigenvalues is given by

$$(z-u_1)(z-u_2)\cdots(z-u_n)=z^n+\alpha_1 z^{n-1}+\cdots+\alpha_{n-1}z+\alpha_n=0$$

$$\begin{aligned}\alpha_1 &= (a_1+\delta_1)\\ \alpha_2 &= (a_2+\delta_2)\\ &\vdots\\ \alpha_n &= (a_n+\delta_n)\end{aligned}$$

Hence, from the equation we have

$$\begin{aligned}K &= \hat{K}T^{-1}\\ &= [\delta_n \quad \delta_{n-1} \quad \cdots \quad \delta_1]T^{-1}\\ &= [\alpha_n-a_n \;\vdots\; \alpha_{n-1}-a_{n-1} \;\vdots\; \cdots \;\vdots\; \alpha_1-a_1]T^{-1}\end{aligned}$$

Ackermann's Formula

By using the state feedback $u(k)=-Kx(k)$, we wish to place the closed loop poles at $u_1,u_2,\cdots u_n$.
That is, we desire the characteristic equation to be

$|(ZI-G+HK)|=(z-u_1)(z-u_2)\cdots(z-u_n)=z^n+\alpha_1 z^{n-1}+\cdots+\alpha_{n-1}z+\alpha_n=0$

Let us define $\hat{G}=G-HK$

Since Cayley-hamilton theorem states that $\hat{G}$ satisfies its own characteristic equation, we have

$\Phi(\hat{G})=\hat{G}^n+\alpha_1\hat{G}^{n-1}+\cdots+\alpha_{n-1}\hat{G}+\alpha_n I=0$

We now use this equation to derive ackerman's formula.

$I=I$

$\hat{G}=G-HK$

$\hat{G}^2=(G-HK)(G-HK)=G^2-GHK-HK\hat{G}$

$\hat{G}^3=(G-HK)(G-HK)(G-HK)=(G-HK)(G^2-GHK-HK\hat{G})=G^3-G^2HK-GHK\hat{G}-HK\hat{G}^2$

$\vdots$

$\hat{G}^n=(G-HK)(G-HK)\cdots(G-HK)=G^n-G^{n-1}HK-\cdots GHK\hat{G}^{n-2}-HK\hat{G}^{n-1}$

Multiply the above equations in order by $\alpha_n,\alpha_{n-1},\cdots\alpha_1,1$ and adding the results

Left side is $\Phi(\hat{G})$=0.

Right side can be written as

$$\Phi(G) - a_{n-1}HK - a_{n-2}GHK - a_{n-2}HK\widehat{G} - \cdots HK\widehat{G}^{n-1} - G^{n-1}HK$$

$$= \Phi(G) - \begin{bmatrix} H & \vdots & GH & \vdots & \cdots & \vdots & G^{n-1}H \end{bmatrix} \begin{bmatrix} a_{n-1}K + a_{n-2}K\widehat{G} + \cdots + K\widehat{G}^{n-1} \\ a_{n-2}K + a_{n-3}K\widehat{G} + \cdots + K\widehat{G}^{n-2} \\ \vdots \\ K \end{bmatrix} = 0$$

Since the system is completely state controllable, the controllability matrix $\begin{bmatrix} H & \vdots & GH & \vdots & \cdots & \vdots & G^{n-1}H \end{bmatrix}$ is of rank n and its inverse exists.

We can have

$$\begin{bmatrix} a_{n-1}K + a_{n-2}K\widehat{G} + \cdots + K\widehat{G}^{n-1} \\ a_{n-2}K + a_{n-3}K\widehat{G} + \cdots + K\widehat{G}^{n-2} \\ \vdots \\ K \end{bmatrix} = \begin{bmatrix} H & \vdots & GH & \vdots & \cdots & \vdots & G^{n-1}H \end{bmatrix}^{-1} \Phi(G)$$

Premultiplying both side of the equation by $\begin{bmatrix} 0 & \vdots & 0 & \vdots & \cdots & \vdots & 1 \end{bmatrix}$

We have

$$K = \begin{bmatrix} 0 & \vdots & 0 & \vdots & \cdots & \vdots & 1 \end{bmatrix} \begin{bmatrix} H & \vdots & GH & \vdots & \cdots & \vdots & G^{n-1}H \end{bmatrix}^{-1} \Phi(G) \qquad \text{6-19}$$

Above equation is the ackerman's formula to determine the feedback K.

Once the desired characteristic equation is selected, there are several different ways to determine the corresponding state feedback matrix K for a completely controllable system.

Method 1 :

$$\begin{aligned} K &= \widehat{K}T^{-1} \\ &= \begin{bmatrix} \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{bmatrix} T^{-1} \\ &= \begin{bmatrix} \alpha_n - a_n & \vdots & \alpha_{n-1} - a_{n-1} & \vdots & \cdots & \vdots & \alpha_1 - a_1 \end{bmatrix} T^{-1} \end{aligned}$$

Where

The characteristic equation of the original system is

$$|(ZI - G)| = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n = 0$$

The characteristic equation with the desired eigenvalues is given by

$$(z - u_1)(z - u_2)\cdots(z - u_n) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1}z + \alpha_n = 0$$

$T = MW$, where $M = \left[H \ \vdots \ GH \ \vdots \ \cdots \ \vdots \ G^{n-1}H\right]$, which is of rank n,
And

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Method 2 : The desired state feedback gain matrix K can be given by Ackermann's formula.
We have

$$K = \left[0 \ \vdots \ 0 \ \vdots \ \cdots \ \vdots \ 1\right]\left[H \ \vdots \ GH \ \vdots \ \cdots \ \vdots \ G^{n-1}H\right]^{-1}\Phi(G)$$

Method 3:
If the desired eigenvalues $u_1, u_2, \cdots u_n$ are distinct, then the desired state feedback gain matrix K can be given as follows:

$$K = \left[1 \ \vdots \ 1 \ \vdots \ \cdots \ \vdots \ 1\right]\left[\xi_1 \ \vdots \ \xi_2 \ \vdots \ \cdots \ \vdots \ \xi_n\right]^{-1}$$

Where $\left[\xi_1 \ \vdots \ \xi_2 \ \vdots \ \cdots \ \vdots \ \xi_n\right]$ satisfy the equation

$(G - HK)\xi_i = u_i\xi_i,\ i = 1,2,\cdots n$, since they are the eigenvectors of matrix $(G - HK)$

$$\xi_i = (G - u_i I)^{-1} H\ , \quad i = 1,2,\cdots n$$

Special case: for deadbeat response, $u_1, u_2, \cdots u_n$ =0
K is simplified into

$$K = \left[1 \ \vdots \ 0 \ \vdots \ \cdots \ \vdots \ 0\right]\left[\xi_1 \ \vdots \ \xi_2 \ \vdots \ \cdots \ \vdots \ \xi_n\right]^{-1}$$

$$\xi_i = G^{-i} H\ , \quad i = 1,2,\cdots n$$

Method 4: if the order of the system is low, substitute K into the characteristic equation.
$|(ZI - G + HK)|$ =0 and then matches the coefficients of powers in z of this characteristic equation with equal powers in z of the desired characteristic equations.

Example 6.4 Consider the system defined by

$$x(k+1) = Gx(k) + Hu(k)$$

Where $G = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $H = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$,
Note that

$$|(ZI - G)| = \begin{vmatrix} z-1 & -1 \\ 1 & z-1 \end{vmatrix} = z^2 - 2z + 2$$

Hence $a_1 = -2,\ a_2 = 2$

Determine a suitable feedback gain matrix such that the system will have the closed loop pole at $z = 0.5 + j0.5,\ z = 0.5 - j0.5,$

Method 1:

$$M = [H \vdots GH] = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}, \textit{full rank} \text{ controllable}$$

$$W = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \textit{full rank}$$

$$T = MW = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix}$$

The desired characteristic equation for the desired system is

$$|(ZI - G)| = (z - 0.5 - j0.5)(z - 0.5 + j0.5) = z^2 - z + 0.5$$

$$\alpha_1 = -1,\ \alpha_2 = 0.5$$

$$K = [\alpha_2 - a_2 \vdots \alpha_1 - a_1]T^{-1} = [0.5 - 2 \vdots -1 - (-2)]\begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix}^{-1} = [-1.5 \quad 1]\begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix} = [-0.25 \quad 0.5]$$

Method 2:

Referring to Ackermann's formula

$$\begin{aligned} K &= [0 \ \vdots \ 0 \ \vdots \ \cdots \ \vdots \ 1][H \ \vdots \ GH \ \vdots \ \cdots \ \vdots \ G^{n-1}H]^{-1}\Phi(G) \\ &= [0 \quad 1][H \quad GH]^{-1}\Phi(G) \\ &= [0 \quad 1]\begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}^{-1}\Phi(G) \\ &= [0 \quad 1]\begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0 \end{bmatrix}\Phi(G) \\ &= [0.5 \quad 0]\Phi(G) \end{aligned}$$

$$\Phi(G) = G^2 - G + 0.5I = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^2 - \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} + 0.5I = \begin{bmatrix} -0.5 & 1 \\ -1 & -0.5 \end{bmatrix}$$

$$K = [0.5 \quad 0]\Phi(G) = [0.5 \quad 0]\begin{bmatrix} -0.5 & 1 \\ -1 & -0.5 \end{bmatrix} = [-0.25 \quad 0.5]$$

Method 3:

$$K = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix}^{-1}$$

$$\xi_i = \left(G - u_i I\right)^{-1} H \text{ ,} \quad i = 1,2$$

$$\begin{aligned}
\xi_1 &= \left(G - u_1 I\right)^{-1} H \\
&= \left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - \left(0.5 + j0.5\right) I \right)^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 0.5 - j0.5 & 1 \\ -1 & 0.5 - j0.5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
&= \frac{1}{1 - 0.5j} \begin{bmatrix} -2 \\ 1 - j \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\xi_2 &= \left(G - u_2 I\right)^{-1} H \\
&= \left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - \left(0.5 - j0.5\right) I \right)^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 0.5 + j0.5 & 1 \\ -1 & 0.5 + j0.5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
&= \frac{1}{1 + 0.5j} \begin{bmatrix} -2 \\ 1 + j \end{bmatrix}
\end{aligned}$$

$$\begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{-2}{1 - 0.5j} & \dfrac{-2}{1 + 0.5j} \\ \dfrac{1 - j}{1 - 0.5j} & \dfrac{1 + j}{1 + 0.5j} \end{bmatrix}^{-1} = \frac{1}{-3.2j} \begin{bmatrix} \dfrac{1 + j}{1 + 0.5j} & -\dfrac{-2}{1 + 0.5j} \\ -\dfrac{1 - j}{1 - 0.5j} & \dfrac{-2}{1 - 0.5j} \end{bmatrix}$$

$$\begin{aligned}
K &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{1}{-3.2j} \begin{bmatrix} \dfrac{1 + j}{1 + 0.5j} & -\dfrac{-2}{1 + 0.5j} \\ -\dfrac{1 - j}{1 - 0.5j} & \dfrac{-2}{1 - 0.5j} \end{bmatrix} = \begin{bmatrix} -0.25 & 0.5 \end{bmatrix}
\end{aligned}$$

Method 4:

For lower order system, it will be simpler to substitute the K into the characteristic equation.

$$\left|(ZI-G+HK)\right| = \left|\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right|$$

$$= \left|\begin{bmatrix} z-1 & -1 \\ 1+2k_1 & z-1+2k_2 \end{bmatrix}\right|$$

$$= z^2 + (2k_2 - 2)z + 2 - 2k_2 + 2k_1 = 0$$

The desired characteristic equation for the desired system is

$$\left|(ZI-G)\right| = (z-0.5-j0.5)(z-0.5+j0.5) = z^2 - z + 0.5 = 0$$

Thus: $(2k_2 - 2) = -1$ and $2 - 2k_2 + 2k_1 = 0.5$

$k_1 = -0.25$, $k_2 = 0.5$

Deadbeat Response.

Consider the system defined by

$x(k+1) = Gx(k) + Hu(k)$

With state feedback $u(k) = -Kx(k)$

The state equation becomes

$x(k+1) = (G-HK)x(k)$

Note: the solution of the last equation is given by

$x(k) = (G-HK)^k x(0)$

If the eigenvalues of $(G-HK)$ lie inside the unit circle, then the system is asymptotically stable.

By choosing all eigenvalues of $(G-HK)$ to be zero, it is possible to get the deadbeat response, or

$x(k) = 0, \quad for \;\; k \geq q, q \leq n$

Nilpotent matrix:

$$N_{n\times n} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \text{ we have } N^n = 0$$

Recall:

$$\left|\left(ZI-G+HK\right)\right| = \left|\left(ZI-\hat{G}+\hat{H}\hat{K}\right)\right|$$

$$= \left| z\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \delta_n & \delta_{n-1} & \delta_{n-2} & \cdots & \delta_1 \end{bmatrix} \right|$$

$$= \left|\begin{bmatrix} z & -1 & 0 & \cdots & 0 \\ 0 & z & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ a_n+\delta_n & a_{n-1}+\delta_{n-1} & a_{n-2}+\delta_{n-2} & \cdots & z+a_1+\delta_1 \end{bmatrix}\right|$$

$$= z^n + \left(a_1+\delta_1\right)z^{n-1} + \cdots + \left(a_{n-1}+\delta_{n-1}\right)z + a_n + \delta_n = 0$$

when $u_1, u_2, \cdots u_n$ =0 we can easily get

$$\left|\left(\hat{G}-\hat{H}\hat{K}\right)\right|$$

$$= \left|\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \delta_n & \delta_{n-1} & \delta_{n-2} & \cdots & \delta_1 \end{bmatrix}\right|$$

$$= \left|\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\right|$$

Which is a nilpotent matrix.

Thus we have $\left(\hat{G}-\hat{H}\hat{K}\right)^n = 0$

In terms of original state , we have

$$x(n) = \left(G-HK\right)^n x(0) = \left(T\hat{G}T^{-1} - T\hat{H}K\right)^n x(0) = \left(T\left(\hat{G}-\hat{H}KT\right)T^{-1}\right)^n x(0) = \left(T\left(\hat{G}-\hat{H}KT\right)T^{-1}\right)^n x(0)$$

$$= \left(T\left(\hat{G}-\hat{H}\hat{K}\right)^n T^{-1}\right)x(0) = 0$$

Remarks:
If the desired eigenvalues are all zeros then any initial state $x(0)$ can be brought to the origin in at most n sampling periods and the response is deadbeat, provided the control signal $u(k)$ is unbounded.
In deadbeat response, the sampling period is the only design parameter. The designer must choose the sampling period carefully so that an extremely large control magnitude is not required in normal operation of the system.
Trade off must be made between the magnitude of the control signal and the response speed.

Example 6.5 Consider the system defined by

$$x(k+1) = Gx(k) + Hu(k)$$

Where $G = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \; H = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$

Note that

$$|(ZI - G)| = \begin{vmatrix} z-1 & -1 \\ 1 & z-1 \end{vmatrix} = z^2 - 2z + 2$$

Hence $a_1 = -2, \; a_2 = 2$

Determine a suitable feedback gain matrix such that the system will have the closed loop pole at $z = 0 \; z = 0$, which is dead beat response.

$$T = MW = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix}$$

The desired characteristic equation for the desired system is

$$\alpha_1 = 0, \; \alpha_2 = 0$$

$$K = [\alpha_2 - a_2 \vdots \alpha_1 - a_1]T^{-1} = [0 - 2 \vdots 0 - (-2)] \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix}^{-1} = [-2 \quad 2] \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix} = [0 \quad 1]$$

$$\begin{bmatrix} \hat{x}_1(k+1) \\ \hat{x}_2(k+1) \end{bmatrix} = T^{-1} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} T \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} - T^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} [0 \quad 1] T \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} - \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix}$$

For any initial state given by $\begin{bmatrix} \hat{x}_1(0) \\ \hat{x}_2(0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\begin{bmatrix} \hat{x}_1(1) \\ \hat{x}_2(1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} \hat{x}_1(2) \\ \hat{x}_2(2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus the state $\hat{X}(k)$ for k=2,3,4… becomes zero and the response is indeed deadbeat.

Control system with reference Input:

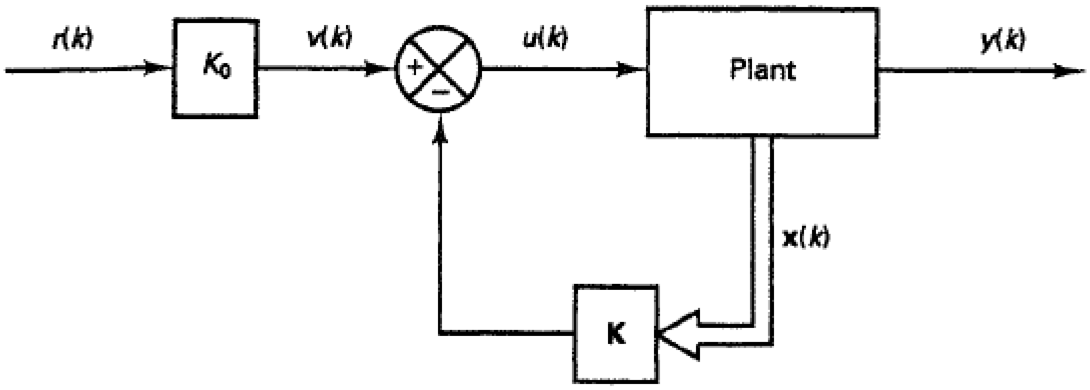


Figure 6.3

Consider the system in above figure.

$$x(k+1) = Gx(k) + Hu(k)$$
$$y(k) = Cx(k)$$

The control signal $u(k)$ is given by

$$u(k) = K_0 r(k) - Kx(k)$$

By eliminating $u(k)$ from the state equation, we have

$$x(k+1) = Gx(k) + H(K_0 r(k) - Kx(k)) = (G - HK)x(k) + HK_0 r(k)$$

The characteristic equation for the system is

$$|(ZI - G + HK)| = 0$$

If the system is completely state controllable, then the feedback gain matrix K can be determined to yield the desired closed-loop poles.

Remarks: the state feedback can change the characteristic equation for the system, it also changes the steady state gain of the entire system. K_0 can be adjusted such that the unit-step response of the system at steady state is unity.

VI.6. State Observers

Note:

1) In practice, not all state variables are available for direct measurement.
2) In many practical cases, only a few state variables of a given system are measurable.
3) Hence, it is necessary to estimate the state variables that are not directly measurable. Such estimation is commonly called observation.

A state observer, also called a state estimator, is a subsystem in the control system that performs an estimation of the state variables based on the measurements of the output and control variables.

Full order state observation: estimate all n state variables regardless of whether some state variables are available for direct measurement.
Minimum order state observation: observation of only the unmeasurable state variables.
Reduced order state observation: observation of all unmeasurable state variables plus some of the measurable state variables.

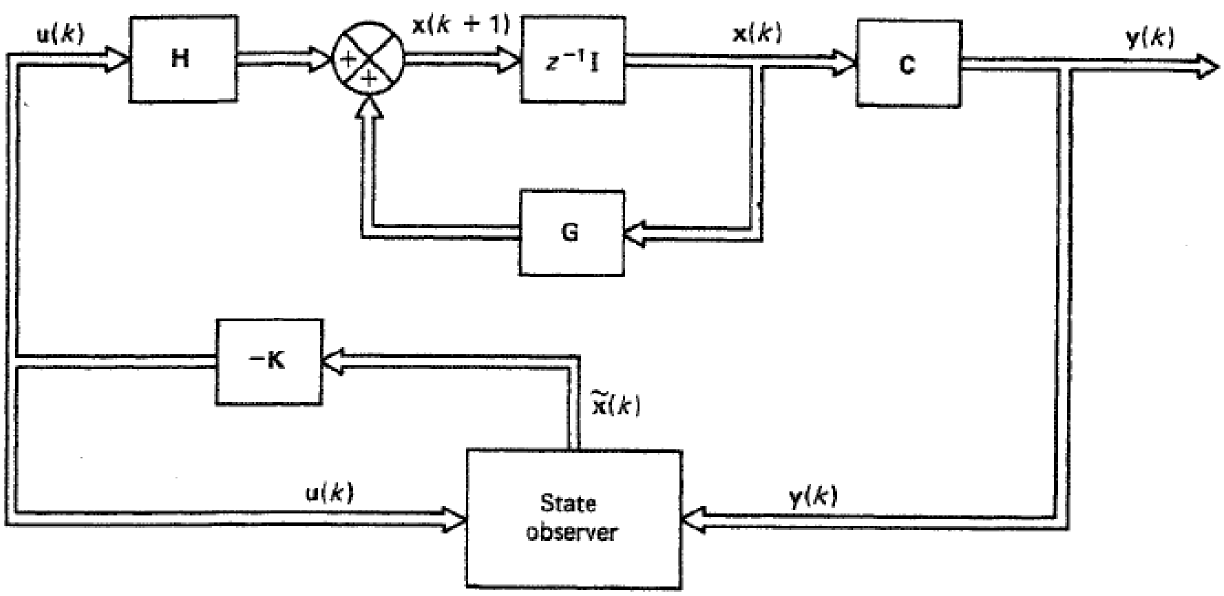


Figure 6.4

Necessary and sufficient condition for state Observation

The state equation is $x(k+1) = Gx(k) + Hu(k)$ 6-20

$$y(k) = Cx(k) \quad \text{6-21}$$

Where

$x(k)$ state vector at kth sampling instant
$u(k)$ control vector at kth sampling instant
$y(k)$ output vector (m-vector)
G $n \times n$ non singular matrix
H $n \times r$ matrix
C $m \times n$ matrix

From 6-20, we have

$$G^{-1}x(k+1) = x(k) + G^{-1}Hu(k)$$

$$\Rightarrow x(k) = G^{-1}x(k+1) - G^{-1}Hu(k) \qquad \text{6-22}$$

Shifting k by 1,

$$x(k-1) = G^{-1}x(k) - G^{-1}Hu(k-1) = G^{-1}\left(G^{-1}x(k+1) - G^{-1}Hu(k)\right) - G^{-1}Hu(k-1)$$

$$= G^{-2}x(k+1) - G^{-2}Hu(k) - G^{-1}Hu(k-1)$$

Similarly

$$x(k-2) = G^{-3}x(k+1) - G^{-3}Hu(k) - G^{-2}Hu(k-1) - G^{-1}Hu(k-2)$$

$$\vdots$$

$$x(k-n+1) = G^{-n}x(k+1) - G^{-n}Hu(k) - G^{-n+1}Hu(k-1) - \cdots G^{-1}Hu(k-n+1)$$

Substitute 6-22 into 6-21, we have

$$y(k) = Cx(k) = CG^{-1}x(k+1) - CG^{-1}Hu(k)$$

$$y(k-1) = Cx(k-1) = CG^{-2}x(k+1) - CG^{-2}Hu(k) - CG^{-1}Hu(k-1)$$

$$y(k-2) = Cx(k-2) = CG^{-3}x(k+1) - CG^{-3}Hu(k) - CG^{-2}Hu(k-1) - CG^{-1}Hu(k-2)$$

$$\vdots$$

$$y(k-n+1) = Cx(k-n+1) = CG^{-n}x(k+1) - CG^{-n}Hu(k) - CG^{-n+1}Hu(k-1) - \cdots CG^{-1}Hu(k-n+1)$$

Combining them

$$\begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-n+1) \end{bmatrix} = \begin{bmatrix} CG^{-1} \\ CG^{-2} \\ \vdots \\ CG^{-n} \end{bmatrix} x(k+1) - \begin{bmatrix} CG^{-1}H & 0 & \cdots & 0 \\ CG^{-2}H & CG^{-1}H & \cdots & 0 \\ \vdots & \vdots & CG^{-1}H & \vdots \\ CG^{-n}H & CG^{n-1}H & \cdots & CG^{-1}H \end{bmatrix} \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(k-n+1) \end{bmatrix}$$

Or

$$\begin{bmatrix} CG^{-1} \\ CG^{-2} \\ \vdots \\ CG^{-n} \end{bmatrix} x(k+1) = \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-n+1) \end{bmatrix} + \begin{bmatrix} CG^{-1}H & 0 & \cdots & 0 \\ CG^{-2}H & CG^{-1}H & \cdots & 0 \\ \vdots & \vdots & CG^{-1}H & \vdots \\ CG^{-n}H & CG^{n-1}H & \cdots & CG^{-1}H \end{bmatrix} \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(k-n+1) \end{bmatrix}$$

Note the right side of the equation is entirely known. Hence $x(k+1)$ can be determined if and only if

$\begin{bmatrix} CG^{-1} \\ CG^{-2} \\ \vdots \\ CG^{-n} \end{bmatrix}$ is full rank. Or $\begin{bmatrix} CG^{n-1} \\ CG^{n-2} \\ \vdots \\ C \end{bmatrix}$ is full rank since G is not singular

Or $\begin{bmatrix} C^* & G^*C^* & (G^2)^*C^* & (G^*)^{n-1}C^* \end{bmatrix}$ is of full rank.

If $y(k)$ is scalar, and matrix C is a 1 by n matrix. Then $x(k+1)$ can be obtained

$$x(k+1) = \begin{bmatrix} CG^{-1} \\ CG^{-2} \\ \vdots \\ CG^{-n} \end{bmatrix}^{-1} \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-n+1) \end{bmatrix} + \begin{bmatrix} CG^{-1} \\ CG^{-2} \\ \vdots \\ CG^{-n} \end{bmatrix}^{-1} \begin{bmatrix} CG^{-1}H & 0 & \cdots & 0 \\ CG^{-2}H & CG^{-1}H & \cdots & 0 \\ \vdots & \vdots & CG^{-1}H & \vdots \\ CG^{-n}H & CG^{n-1}H & \cdots & CG^{-1}H \end{bmatrix} \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(k-n+1) \end{bmatrix}$$

Remarks:

1) $x(k+1)$ can be determined provided the system is completely observable.

2) in the presence of disturbance and measurement noise, $x(k+1)$ can not be estimated accurately.

3) If C is not 1 by n matrix but is a m by n matrix, then the inverse of the matrix $\begin{bmatrix} CG^{-1} \\ CG^{-2} \\ \vdots \\ CG^{-n} \end{bmatrix}$ is not defined. To cope with this, we will use dynamic model.

Consider the system

is $x(k+1) = Gx(k) + Hu(k)$ 6-23

$y(k) = Cx(k)$ 6-24

We assume that the state $x(k)$ is to be approximated by the state $\tilde{x}(k)$ of the dynamic model

$\tilde{x}(k+1) = G\tilde{x}(k) + Hu(k)$ 6-25

$\tilde{y}(k) = C\tilde{x}(k)$ 6-26

Where matrices G, H, and C are the same as those original system.
Let us assume that the dynamic model is subjected to the same control signal $u(k)$ as the original system.

If initial conditions are same for the original one as the dynamic one, then $\tilde{x}(k)$ and $x(k)$ will be same. Otherwise, $\tilde{x}(k)$ and $x(k)$ will be different.

If the matrix G is a stable one, however, $\tilde{x}(k)$ will approach $x(k)$ even for different initial conditions.

If we denote the difference between $\tilde{x}(k)$ and $x(k)$ as $e(k) = x(k) - \tilde{x}(k)$

Then subtract 6-25 by 6-20, we obtain
$x(k+1) - \tilde{x}(k+1) = G(x(k) - \tilde{x}(k))$, or $e(k+1) = Ge(k)$
$e(k)$ will approach 0 if G stable.

Remarks: although the state $x(k)$ may not be measurable the output $y(k)$ is measurable.
The dynamic model does not use the measured output $y(k)$.
The dynamic model of equation 6-25 is modified into the following form:
$\tilde{x}(k+1) = G\tilde{x}(k) + Hu(k) + Ke[y(k) - C\tilde{x}(k)]$, where matrix Ke serves as a weighting matrix.

Full-Order State Observer

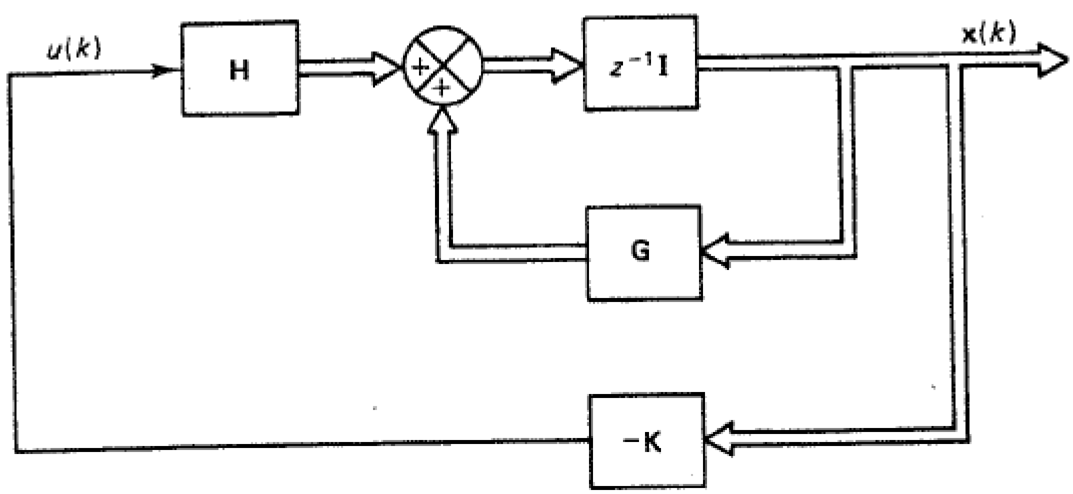


Figure 6.5

Consider the state feedback control system above.

The state equation is $x(k+1) = Gx(k) + Hu(k)$ 6-27

$$y(k) = Cx(k) \qquad \text{6-28}$$
$$u(k) = -Kx(k)$$

Where

$x(k)$ state vector at kth sampling instant
$u(k)$ control vector at kth sampling instant
$y(k)$ output vector (m-vector)
G $n \times n$ non singular matrix
H $n \times r$ matrix
C $m \times n$ matrix
K state feedback gain matrix

We assume that the system is completely state controllable and completely observable, but $x(k)$ is not available for direct measurement.
Following figure shows a state observer incorporated into the system of previous figure.

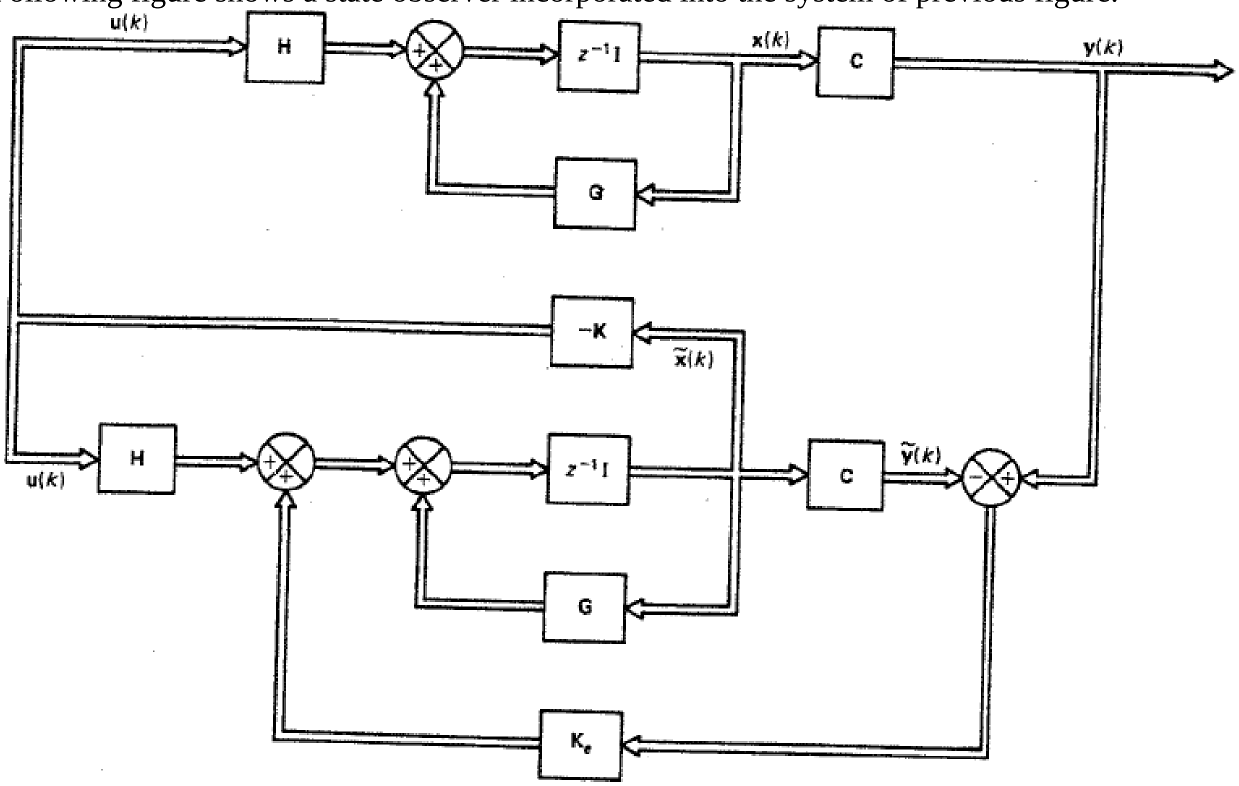


Figure 6.6

$u(k) = -K\tilde{x}(k)$
From above figure we have
$$\tilde{x}(k+1) = G\tilde{x}(k) + Hu(k) + Ke[y(k) - C\tilde{x}(k)] \qquad \text{6-29}$$
Which can be rewritten into

$$\tilde{x}(k+1) = (G - K_e C)\tilde{x}(k) + Hu(k) + K_e y(k) \qquad \text{6-30}$$

Equation 6-30 is called a prediction observer since the estimate $\tilde{x}(k+1)$ is one sampling period ahead of the measurement $y(k)$.

Error Dynamics of the full order state observer

Notice that if $\tilde{x}(k) = x(k)$

$\tilde{x}(k+1) = G\tilde{x}(k) + Hu(k)$

Which is identical to the state equation of the system.

To obtain the observer error equation, let us subtract equation 6-30 from 6-27.

$x(k+1) - \tilde{x}(k+1) = (G - K_e C)(x(k) - \tilde{x}(k))$

Define $e(k) = x(k) - \tilde{x}(k)$

$$e(k+1) = (G - K_e C)e(k) \qquad \text{6-31}$$

From 6-31, we see that the dynamic behavior of the error signal is determined by the eigenvalues of $(G - K_e C)$. If $(G - K_e C)$ is a stable matrix, the error vector will coverge to zero for any initial error $e(0)$. One way to obtain the fast response is to use deadbeat response.

Remarks: 6-20, 6-21 are assumed to be completely observable, an arbitrary placement of the eigenvalues of $(G - K_e C)$ is possible. Notice the eigenvalue of $(G - K_e C)$ and $(G^* - C^* K_e^{\ *})$ are the same.

By use principle of duality, the condition for complete observability for the system defined by

$$x(k+1) = Gx(k) + Hu(k) \qquad \text{6-32}$$

$$y(k) = Cx(k) \qquad \text{6-33}$$

Is same as the complete state controllability condition for the system

$x(k+1) = G * x(k) + C * u(k)$, for this system by selecting a set of n desired eigenvalues of $(G^* - C^* K)$, the state feedback gain matrix K may be determined. The desired matrix K_e, such that the eigenvalues of $(G - K_e C)$ are the same as those of $(G^* - C^* K)$.

$K_e = K *$

Example 6.6 Consider the system defined by

$x(k+1) = Gx(k) + Hu(k)$

$y(k) = Cx(k)$

Where $G = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $H = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 2 \end{bmatrix}$,

Design a full order state observer, assuming that the system configuration is identical to the in above figure. The desired eigenvalues of the observer matrix are $z = 0.5 + j0.5,\ z = 0.5 - j0.5,$ and desired ch equation is

$z^2 - z + 0.5 = 0$

The rank of observability matrix $\left[C* \ \vdots \ G*C*\right] = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$ is 2, thus system is completely observable.

Let us denote the observer feedback gain matrix $K_e = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

Thee characteristic equation of the observer becomes

$$|zI - G + K_eC| = \left| z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}\begin{bmatrix} 0 & 2 \end{bmatrix} \right| = \left\| \begin{bmatrix} z-1 & 1+2k_1 \\ -1 & z-1+2k_2 \end{bmatrix} \right\| = 0$$

$z^2 + (2k_2 - 2)z + 2 + 2k_1 - 2k_2 = 0$

Match with the desired ch equation

$z^2 - z + 0.5 = 0$

We have $K_e = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -0.25 \\ 0.5 \end{bmatrix}$

Note the dual relationship exists between this example and example 6.4. The state feedback gain matrix K obtained in example 6.4 is . $K = \begin{bmatrix} -0.25 & 0.5 \end{bmatrix}$. The observer feedback gain matrix $K_e = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -0.25 \\ 0.5 \end{bmatrix}$ is related to matrix K by the relationship $K_e = K^*$

Design of general prediction observers

Consider the system defined by

$$x(k+1) = Gx(k) + Hu(k) \qquad 6\text{-}34$$

$$y(k) = Cx(k) \qquad 6\text{-}35$$

Where

$x(k)$ state vector

$u(k)$ control vector

$y(k)$ output vector

G $n \times n$ non singular matrix

H $n \times r$ matrix

C $1 \times n$ matrix

The system is assumed to be completely state controllable and completely observable. Thus the inverse of $\left[C* \quad G*C* \quad (G^2)*C* \quad (G*)^{n-1}C*\right]$ exists

Assume the control law is $u(k) = -K\tilde{x}(k)$, where $\tilde{x}(k)$ is the observed state and K is an $r \times n$ matrix.

Assume the system configuration is in figure 6.6.

$$\begin{aligned}\tilde{x}(k+1) &= G\tilde{x}(k) + Hu(k) + Ke[y(k) - C\tilde{x}(k)] \\ &= (G - KeC)\tilde{x}(k) + Hu(k) + KeCx(k)\end{aligned} \qquad \text{6-36}$$

Define $Q = (WN^*)^{-1}$, where $N = [C* \quad G*C* \quad (G^2)*C* \quad (G*)^{n-1}C*]$

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Where $|(ZI - G)| = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n = 0$ are the ch equation of the original state equation given by 6-34.

Next define $x(k) = Q\xi(k)$ 6-37

$$\xi(k+1) = Q^{-1}GQ\xi(k) + Q^{-1}Hu(k) \qquad \text{6-38}$$

$$y(k) = CQ\xi(k) \qquad \text{6-39}$$

Where, $\hat{G} = Q^{-1}GQ$ *and* $\hat{H} = Q^{-1}H$, $\hat{C} = CQ$, $\hat{D} = D$, or

$$Q^{-1}GQ = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & & 0 & -a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \qquad \text{6-40}$$

$$CQ = [0 \quad 0 \quad \cdots \quad 0 \quad 1] \qquad \text{6-41}$$

Now define $\tilde{x}(k) = Q\tilde{\xi}(k)$

$$\begin{aligned}\tilde{x}(k+1) &= G\tilde{x}(k) + Hu(k) + Ke[y(k) - C\tilde{x}(k)] \\ &= (G - KeC)\tilde{x}(k) + Hu(k) + KeCx(k)\end{aligned}$$ will be changed into

$$\tilde{\xi}(k+1) = Q^{-1}(G - KeC)Q\tilde{\xi}(k) + Q^{-1}Hu(k) + Q^{-1}KeCQ\xi(k) \qquad \text{6-42}$$

Subtract 6-42 from 6-38, we have

$$\xi(k+1) - \tilde{\xi}(k+1) = (Q^{-1}GQ - Q^{-1}KeCQ)(\xi(k) - \tilde{\xi}(k)) \qquad \text{6-43}$$

Define $e(k) = \xi(k) - \tilde{\xi}(k)$

$$e(k+1) = Q^{-1}(G - KeC)Qe(k) \qquad \text{6-44}$$

Remarks: Select the desired observer poles and then to determine matrix Ke . If we require $e(k)$ to reach zero as soon as possible, then we require the error response to be deadbeat, then all eigenvalues of $(G - KeC)$ must be zero.

Let us write $Q^{-1}Ke = \begin{bmatrix} \delta_n \\ \delta_{n-1} \\ \vdots \\ \delta_1 \end{bmatrix}$, then

$$Q^{-1}KeCQ = \begin{bmatrix} \delta_n \\ \delta_{n-1} \\ \vdots \\ \delta_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \delta_n \\ 0 & 0 & \cdots & 0 & \delta_{n-1} \\ 0 & 0 & \cdots & 0 & \delta_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \delta_1 \end{bmatrix}$$

And

$$Q^{-1}(G - KeC)Q = Q^{-1}GQ - \begin{bmatrix} \delta_n \\ \delta_{n-1} \\ \vdots \\ \delta_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n - \delta_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} - \delta_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} - \delta_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 - \delta_1 \end{bmatrix}$$

The ch equation becomes

$$\left| zI - Q^{-1}(G - KeC)Q \right| = 0$$

$$\begin{vmatrix} z & 0 & \cdots & 0 & a_n + \delta_n \\ -1 & z & \cdots & 0 & a_{n-1} + \delta_{n-1} \\ 0 & -1 & \cdots & 0 & a_{n-2} + \delta_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & z + a_1 + \delta_1 \end{vmatrix} = 0$$

or

$$\left| zI - Q^{-1}(G - KeC)Q \right| = z^n + (a_1 + \delta_1)z^{n-1} + \cdots + (a_{n-1} + \delta_{n-1})z + a_n + \delta_n = 0 \qquad \text{6-45}$$

Suppose the desired characteristic equation for error dynamics is

$$z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1}z + \alpha_n = 0 \qquad \text{6-46}$$

Compare the coefficient of 6-45 and 6-46, we have

$$\alpha_1 = (a_1 + \delta_1)$$
$$\alpha_2 = (a_2 + \delta_2)$$
$$\vdots$$
$$\alpha_n = (a_n + \delta_n)$$

Hence, from the equation we have

$$\delta_1 = \alpha_1 - a_1$$
$$\delta_2 = \alpha_2 - a_2$$
$$\vdots$$
$$\delta_n = \alpha_n - a_n$$

Since $Q^{-1}Ke = \begin{bmatrix} \delta_n \\ \delta_{n-1} \\ \vdots \\ \delta_1 \end{bmatrix}$, we have $Ke = Q \begin{bmatrix} \delta_n \\ \delta_{n-1} \\ \vdots \\ \delta_1 \end{bmatrix} = (WN^*)^{-1} \begin{bmatrix} \delta_n \\ \delta_{n-1} \\ \vdots \\ \delta_1 \end{bmatrix}$

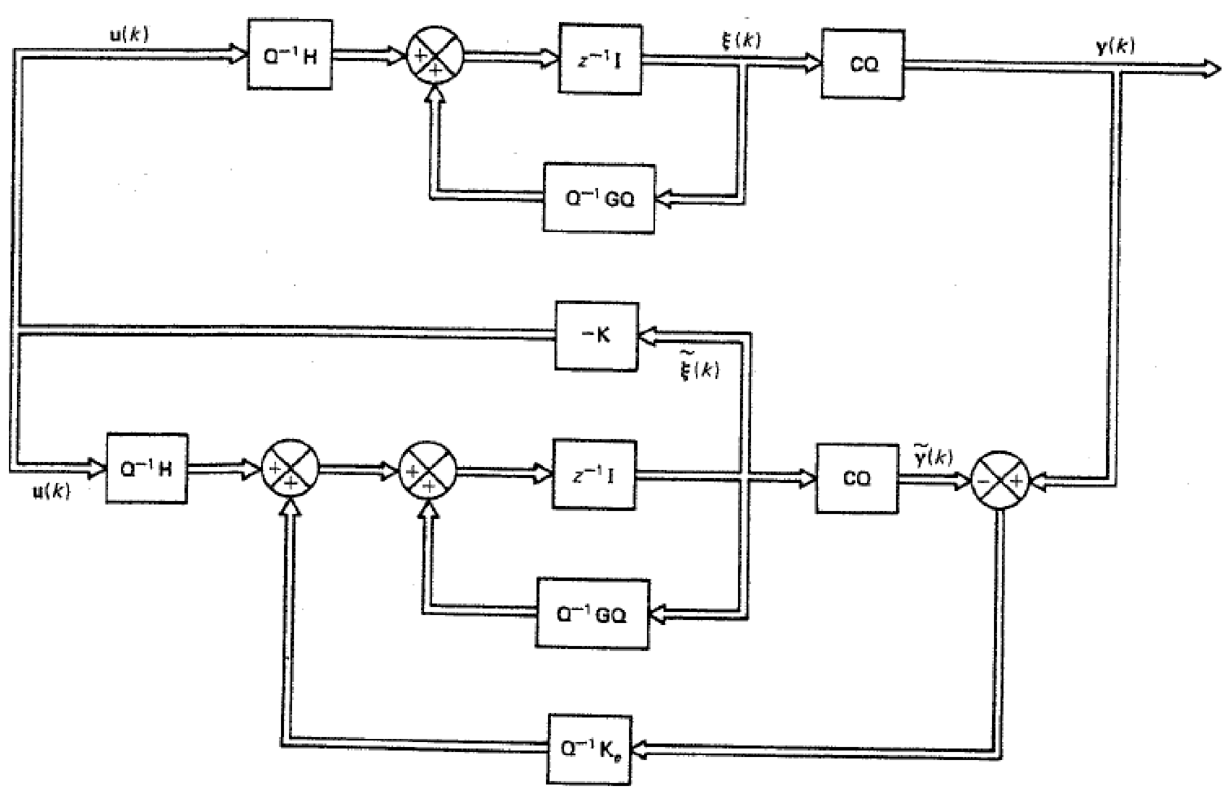


Figure 6.7 Alternative representation of the observed-state feedback control system

For deadbeat response, the desired ch equation becomes $z^n = 0$

$$Ke = Q\begin{bmatrix}\delta_n \\ \delta_{n-1} \\ \vdots \\ \delta_1\end{bmatrix} = \left(WN^*\right)^{-1}\begin{bmatrix}-a_n \\ -a_{n-1} \\ \vdots \\ -a_1\end{bmatrix} \qquad \text{6-47}$$

Ackermann`s Formula : procedure is same as for state feedback.

$$K_e = \Phi(G)\begin{bmatrix}C \\ CG \\ \vdots \\ CG^{n-1}\end{bmatrix}^{-1}\begin{bmatrix}0 \\ 0 \\ \vdots \\ 1\end{bmatrix}$$

Summary: The full order prediction observer is given by equation

$$\begin{aligned}\tilde{x}(k+1) &= G\tilde{x}(k) + Hu(k) + Ke\left[y(k) - C\tilde{x}(k)\right] \\ &= (G - KeC)\tilde{x}(k) + Hu(k) + KeCx(k)\end{aligned}$$

The observed state feedback is given by

$$u(k) = -K\tilde{x}(k)$$

If we have the feedback equation substituted into the observer equation, we obtain

$$\tilde{x}(k+1) = (G - K_eC - HK)\tilde{x}(k) + Hu(k) + K_e y(k)$$

Similar to the state feedback, four methods will be used to determine the observer feedback gain K_e.

Method 1 :

$$K_e = Q\begin{bmatrix}\alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_1 - a_1\end{bmatrix} = (WN*)^{-1}\begin{bmatrix}\alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_1 - a_1\end{bmatrix}$$

Where $Q = \left(WN^*\right)^{-1}$, $N = \left[C* \quad G*C* \quad (G^2)*C* \quad (G*)^{n-1}C*\right]$

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

α_i’s are the coefficients of the desired characteristic equation

$$\left|(ZI - G)\right| = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1}z + \alpha_n = 0$$

The characteristic equation of the original system is

$$\left|(ZI - G)\right| = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

Note: if the system is already in an observable canonical form, then the matrix K_e can be determined easily, because matrix $(WN^*)^{-1}$ becomes an identity matrix, thus $(WN^*)^{-1} = I$

Method 2 : The desired observer feedback gain matrix K_e can be given by Ackermann's formula.
We have

$$K_e = \Phi(G) \begin{bmatrix} C \\ CG \\ \vdots \\ CG^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\Phi(G) = G^n + \alpha_1 G^{n-1} + \cdots + \alpha_{n-1} G + \alpha_n I$$

Method 3:
If the desired eigenvalues $u_1, u_2, \cdots u_n$ of matrix $(G - K_e C)$ are distinct, then the observer feedback gain matrix K_e may be given by the equation as follows:

$$K_e = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Where $\begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix}$ satisfy the equation

$$\eta_i = C(G - u_i I)^{-1}, \quad i = 1,2,\cdots n$$

Special case: for deadbeat response, $u_1, u_2, \cdots u_n = 0$
K_e is simplified into

$$Ke = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\eta_i = CG^{-i}, \quad i = 1,2,\cdots n$$

Method 4: If the order of the system is low, substitute K_e into the characteristic equation. $|(ZI - G + K_eC)|$ =0 and then matches the coefficients of powers in z of this characteristic equation with equal powers in z of the desired characteristic equations.

Example 6.7 Consider the system defined by

$$x(k+1) = Gx(k) + Hu(k)$$
$$y(k) = Cx(k)$$

Where $G = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},\ H = \begin{bmatrix} 0 \\ 2 \end{bmatrix},\ C = \begin{bmatrix} 0 & 2 \end{bmatrix},$

Design a full order state observer, assuming that the system configuration is identical to the in above figure. The desired eigenvalues of the observer matrix are $z = 0.5 + j0.5,\ z = 0.5 - j0.5,$ and desired ch equation is

$$z^2 - z + 0.5 = 0$$
$$|(ZI - G)| = z^2 - 2z + 2 = 0$$

Method 1:

$$W = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix},\ N = \begin{bmatrix} C^* & \vdots & G^*C^* \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$$

$$Q = (WN^*)^{-1} = \left(\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}\right)^{-1} = \left(\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}\right)^{-1} = \left(\begin{bmatrix} 0.5 & 0.5 \\ 0 & 0.5 \end{bmatrix}\right)$$

$$K_e = Q\begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 0.5 - 2 \\ -1 - (-2) \end{bmatrix} = \begin{bmatrix} -0.25 \\ 0.5 \end{bmatrix}$$

Method 2:

$$\Phi(G) = G^2 - G + 0.5I = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^2 - \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + 0.5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$
$$= \begin{bmatrix} -0.5 & -1 \\ 1 & -0.5 \end{bmatrix}$$

$$K_e = \Phi(G)\begin{bmatrix} C \\ CG \\ \vdots \\ CG^{n-1} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 & -1 \\ 1 & -0.5 \end{bmatrix}\begin{bmatrix} C \\ CG \\ \vdots \\ CG^{n-1} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 & -1 \\ 1 & -0.5 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & -1 \\ 1 & -0.5 \end{bmatrix}\begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.25 & -0.25 \\ -0.75 & 0.5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.25 \\ 0.5 \end{bmatrix}$$

Method 3:

$$\eta_1 = C(G - u_1 I)^{-1}$$

$$= \begin{bmatrix} 0 & 2 \end{bmatrix}\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - (0.5 + j0.5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} 0 & 2 \end{bmatrix}\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0.5 + j0.5 & 0 \\ 0 & 0.5 + j0.5 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} 0 & 2 \end{bmatrix}\left(\begin{bmatrix} 0.5 - j0.5 & -1 \\ 1 & 0.5 - j0.5 \end{bmatrix}\right)^{-1}$$

$$= \frac{1}{1 - 0.5j}\begin{bmatrix} -2 & 1 - j \end{bmatrix}$$

$$\eta_2 = C(G - u_2 I)^{-1}$$

$$= \begin{bmatrix} 0 & 2 \end{bmatrix}\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - (0.5 - j0.5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} 0 & 2 \end{bmatrix}\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0.5 - j0.5 & 0 \\ 0 & 0.5 - j0.5 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} 0 & 2 \end{bmatrix}\left(\begin{bmatrix} 0.5 + j0.5 & -1 \\ 1 & 0.5 + j0.5 \end{bmatrix}\right)^{-1}$$

$$== \frac{1}{1 + 0.5j}\begin{bmatrix} -2 & 1 + j \end{bmatrix}$$

$$K_e = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-2}{1-0.5j} & \frac{1-j}{1-0.5j} \\ \frac{-2}{1+0.5j} & \frac{1+j}{1+0.5j} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{-3.2j} \begin{bmatrix} \frac{1+j}{1+0.5j} & \frac{-1+j}{1-0.5j} \\ \frac{2}{1+0.5j} & \frac{-2}{1-0.5j} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{-3.2j} \begin{bmatrix} \frac{1+j}{1+0.5j} + \frac{-1+j}{1-0.5j} \\ \frac{2}{1+0.5j} + \frac{-2}{1-0.5j} \end{bmatrix} = \begin{bmatrix} -0.25 \\ 0.5 \end{bmatrix}$$

Method 4:

The rank of observability matrix $\left[C^* \;\vdots\; G^*C^*\right] = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$ is 2, thus system is completely observable.

Let us denote the observer feedback gain matrix $K_e = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

The characteristic equation of the observer becomes

$$\left|zI - G + K_e C\right| = \left| z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}\begin{bmatrix} 0 & 2 \end{bmatrix} \right| = \left\| \begin{bmatrix} z-1 & 1+2k_1 \\ -1 & z-1+2k_2 \end{bmatrix} \right\| = 0$$

$$z^2 + (2k_2 - 2)z + 2 + 2k_1 - 2k_2 = 0$$

Match with the desired ch equation

$$z^2 - z + 0.5 = 0$$

We have $K_e = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -0.25 \\ 0.5 \end{bmatrix}$

Effects of the addition of the observer on a closed-loop system.

Consider the completely state controllable and completely observable system defined by the equations

$$x(k+1) = Gx(k) + Hu(k)$$
$$y(k) = Cx(k)$$

For the state feedback control based on the observed state $\tilde{x}(k)$, we have

$u(k) = -K\tilde{x}(k)$

$x(k+1) = Gx(k) - HK\tilde{x}(k) = (G - HK)x(k) + HK(x(k) - \tilde{x}(k))$

Define $e(k) = x(k) - \tilde{x}(k)$

$x(k+1) = (G - HK)x(k) + HKe(k)$

Also the observer was given by

$e(k+1) = (G - K_e C)e(k)$

Combine this two:

$$\begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} (G - HK) & HK \\ 0 & G - K_e C \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix}$$

The characteristic equation for the system is

$$\begin{vmatrix} (ZI - G + HK) & -HK \\ 0 & ZI - G + K_e C \end{vmatrix} = 0$$

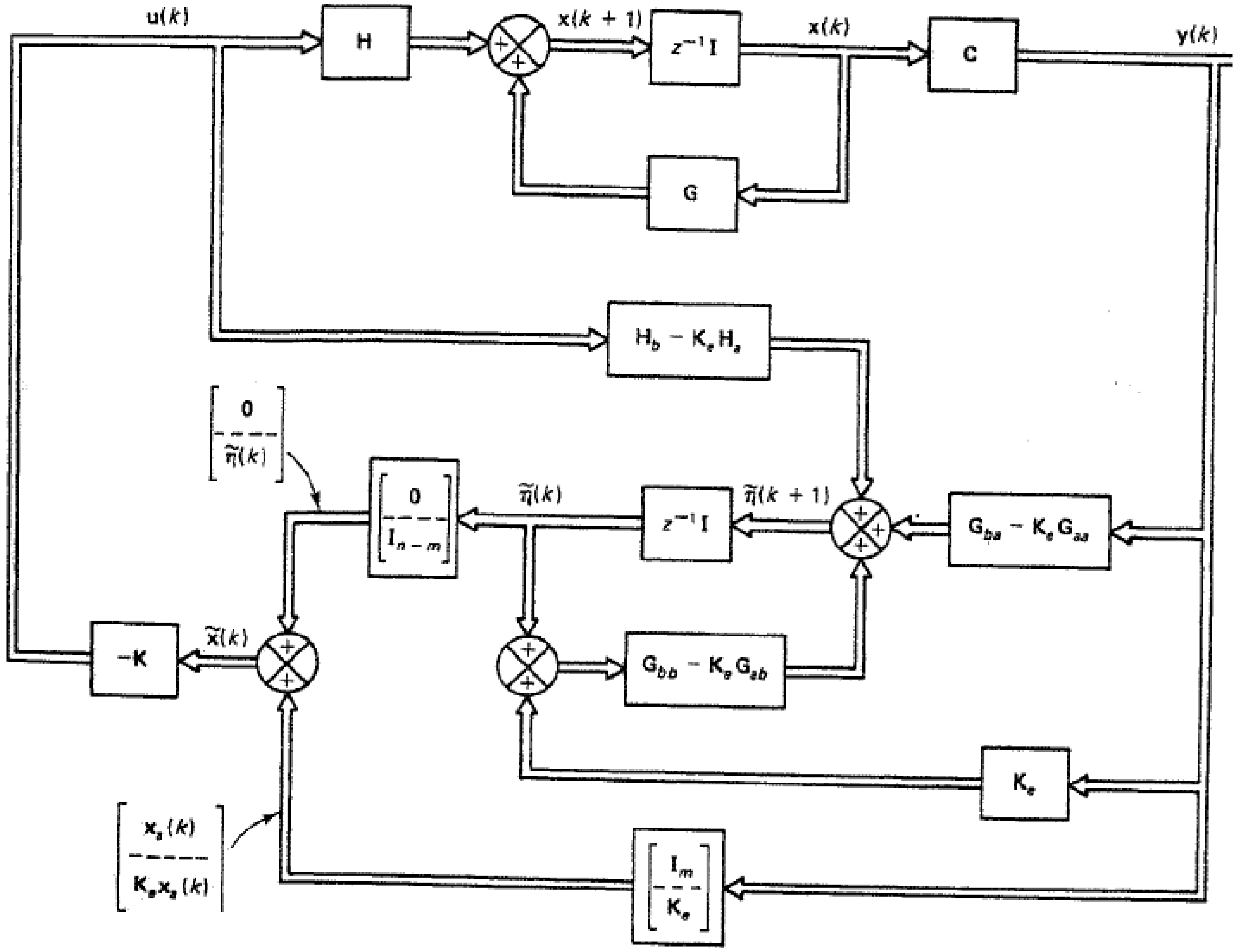
Pole placement design and the observer design are independent of each other.

Remarks: The poles of the observer are usually chosen so that the observer response is much faster than the system response. A rule of thumb is to choose an observer response at least four to five times faster than the system response (or deadbeat response).

Current observer:

In the prediction observer the observed state $\tilde{x}(k)$ is obtained from measurements of the output vector up to $y(k-1)$ and of the control vector up to $u(k-1)$. A different formulation of the state observer is to use $y(k)$ for the estimation of $\tilde{x}(k)$. This can be done in two steps. First step we determine $z(k+1)$, an approximation of $x(k+1)$ based on $\tilde{x}(k)$ and $u(k)$. In the second step, we use $y(k+1)$ to improve $x(k+1)$.

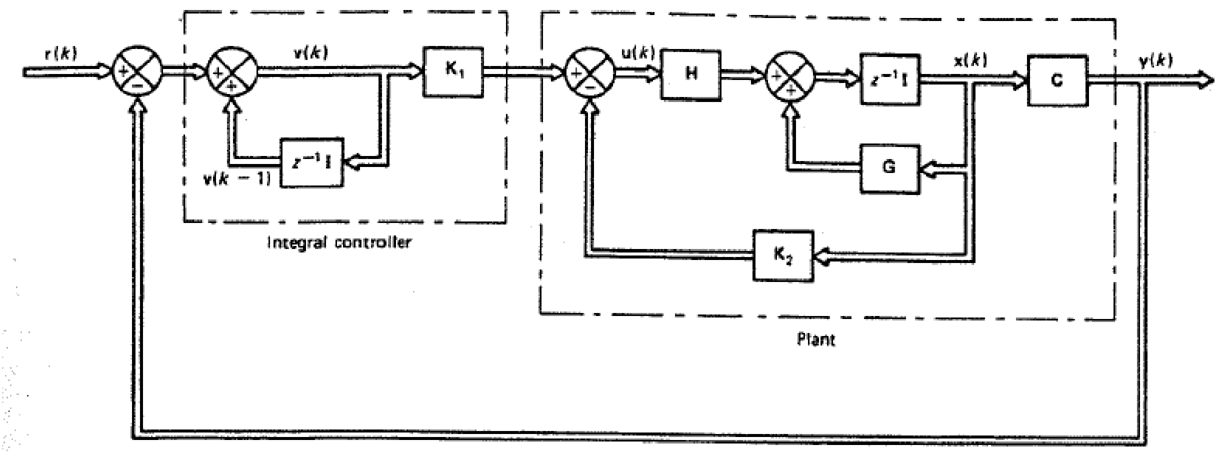
Minimum order observer:



State feedback control scheme where the feedback state consists of the measured portion of the state and the observed portion of the state obtained by use of the minimum order observer.

VI.7. Servo Systems

Note: It is generally required that the system has one or more integrators within the closed loop, to eliminate the steady state error to step inputs.



For the plant:

$$x(k+1) = Gx(k) + Hu(k) \qquad 6\text{-}48$$
$$y(k) = Cx(k) \qquad 6\text{-}49$$

Where

$x(k)$ Plant state vector
$u(k)$ control vector
$y(k)$ output vector
G $n \times n$ non singular matrix
H $n \times m$ matrix
C $m \times n$ matrix

For the integrator :

$$v(k) = v(k-1) + r(k) - y(k) \qquad 6\text{-}50$$

Where
$v(k)$ actuating error vector
$r(k)$ command input vector
6-50 can be rewritten as follows:

$$
\begin{aligned}
v(k+1) &= v(k) + r(k+1) - y(k+1) \\
&= v(k) + r(k+1) - C(Gx(k) + Hu(k)) \\
&= -CGx(k) + v(k) - CHu(k) + r(k+1)
\end{aligned}
\tag{6-51}
$$

The control vector $u(k)$ is given by

$u(k) = -K_2 x(k) + K_1 v(k)$

We then have

$$
\begin{aligned}
u(k+1) &= -K_2 x(k+1) + K_1 v(k+1) \\
&= -K_2 x(k+1) + K_1(-CGx(k) + v(k) - CHu(k) + r(k+1)) \\
&= -K_2(Gx(k) + Hu(k)) + K_1(-CGx(k) + v(k) - CHu(k) + r(k+1)) \\
&= (K_2 - K_2G - K_1CG)x(k) + (I_m - K_2H - K_1CH)u(k) + K_1 r(k+1)
\end{aligned}
\tag{6-52}
$$

Combine with $x(k+1) = Gx(k) + Hu(k)$

We have $\begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} G & H \\ K_2 - K_2G - K_1CG & I_m - K_2H - K_1CH \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} 0 \\ K_1 \end{bmatrix} r(k+1)$ 6-53

Output equation can be written as

$$
y(k) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}
$$

For the step input $r(k) = r$

$$
\begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} G & H \\ K_2 - K_2G - K_1CG & I_m - K_2H - K_1CH \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} 0 \\ K_1 r \end{bmatrix} \tag{6-54}
$$

$$
\begin{bmatrix} x(\infty) \\ u(\infty) \end{bmatrix} = \begin{bmatrix} G & H \\ K_2 - K_2G - K_1CG & I_m - K_2H - K_1CH \end{bmatrix} \begin{bmatrix} x(\infty) \\ u(\infty) \end{bmatrix} + \begin{bmatrix} 0 \\ K_1 r \end{bmatrix} \tag{6-55}
$$

Define the error vector $x_e(k) = x(k) - x(\infty)$, and $u_e(k) = u(k) - u(\infty)$

Subtracting the equation 6-55 from 6-54, we obtain

$$
\begin{aligned}
\begin{bmatrix} x_e(k+1) \\ u_e(k+1) \end{bmatrix} &= \begin{bmatrix} G & H \\ K_2 - K_2G - K_1CG & I_m - K_2H - K_1CH \end{bmatrix} \begin{bmatrix} x_e(k) \\ u_e(k) \end{bmatrix} \\
&= \begin{bmatrix} G & H \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_e(k) \\ u_e(k) \end{bmatrix} + \begin{bmatrix} 0 \\ I_m \end{bmatrix} w(k)
\end{aligned}
\tag{6-56}
$$

Where $w(k) = \begin{bmatrix} K_2 - K_2G - K_1CG & \vdots & I_m - K_2H - K_1CH \end{bmatrix} \begin{bmatrix} x_e(k) \\ u_e(k) \end{bmatrix}$ 6-57

Define

$\xi(k) = \begin{bmatrix} x_e(k) \\ u_e(k) \end{bmatrix}$, n+m vector

$\hat{G} = \begin{bmatrix} G & H \\ 0 & 0 \end{bmatrix}$, $(n+m)\times(n+m)$ matrix

$\hat{H} = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$, $(n+m)\times m$ matrix

$$\hat{K} = -\begin{bmatrix} K_2 - K_2 G - K_1 CG & \vdots & I_m - K_2 H - K_1 CH \end{bmatrix}$$
$$= \begin{bmatrix} K_2 & K_1 \end{bmatrix} \begin{bmatrix} G - I_n & H \\ CG & CH \end{bmatrix} + \begin{bmatrix} 0 & -I_m \end{bmatrix} \qquad , \ m\times(n+m) \text{ matrix}$$

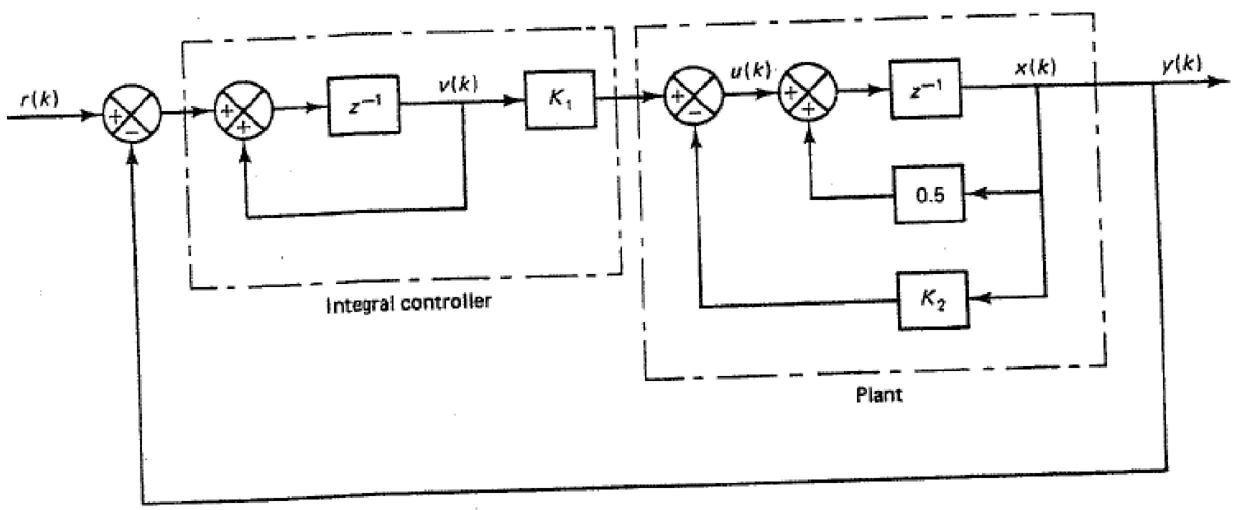
Then we have $\xi(k+1) = \hat{G}\xi(k) + \hat{H}w(k)$ 6-58

And $w(k) = -\hat{K}\xi(k)$

Thus, 6-58 will be completely state controllable, $\hat{K}$ can be designed, and $\begin{bmatrix} K_2 & K_1 \end{bmatrix}$ can be obtained using following equation:

$$\hat{K} = \begin{bmatrix} K_2 & K_1 \end{bmatrix} \begin{bmatrix} G - I_n & H \\ CG & CH \end{bmatrix} + \begin{bmatrix} 0 & -I_m \end{bmatrix} \Rightarrow \begin{bmatrix} K_2 & K_1 \end{bmatrix} \begin{bmatrix} G - I_n & H \\ CG & CH \end{bmatrix} = \hat{K} + \begin{bmatrix} 0 & I_m \end{bmatrix}$$

Example B-6-17 : Figure shows a servo system where the integral controller has a time delay of one sampling period. Determine the feedforward gain K1 and the feedback gain K2 such that the response to the unit step sequence input is deadbeat.



Solution:

The system equations are

$$x(k+1) = 0.5\,x(k) + u(k)$$
$$y(k) = x(k)$$
$$v(k+1) = v(k) + r(k) - y(k)$$
$$u(k) = K_1 v(k) - K_2 x(k)$$

Thus,

$$u(k+1) = -K_2 x(k+1) + K_1 v(k+1)$$
$$= -K_2 \left[0.5\, x(k) + u(k)\right] + K_1 \left[v(k) + r(k) - y(k)\right]$$
$$= (0.5\, K_2 - K_1)x(k) + (1 - K_2)u(k) + K_1 r(k)$$

Hence, the state equation in terms of x and u becomes

$$\begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0.5\, K_2 - K_1 & 1 - K_2 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} 0 \\ K_1 \end{bmatrix} r(k) \qquad (1)$$

and the output equation becomes

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$$

Define

$$x_e(k) = x(k) - x(\infty)$$
$$u_e(k) = u(k) - u(\infty)$$

Then

$$\begin{bmatrix} x_e(k+1) \\ u_e(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0.5\, K_2 - K_1 & 1 - K_2 \end{bmatrix} \begin{bmatrix} x_e(k) \\ u_e(k) \end{bmatrix}$$

The characteristic equation is

$$\begin{vmatrix} z - 0.5 & -1 \\ -0.5\, K_2 + K_1 & z - 1 + K_2 \end{vmatrix}$$
$$= z^2 + (K_2 - 1.5)z + 0.5 - K_2 + K_1 = 0$$

The desired characteristic equation is

$$z^2 = 0$$

Hence we choose $K_1 = 1$ and $K_2 = 1.5$. Thus, the integral gain constant K_1 is

$$K_1 = 1$$

and the state feedback gain constant K_2 is

$$K_2 = 1.5$$

[It is noted that Equation (6-193) must be modified if it is to be applied to this problem, since the configuration of the integral controller is different from that shown in Figure 6-18.]

To determine the output $y(k)$, notice that

$$y(k) = x(k)$$

By substituting $K_1 = 1$, $K_2 = 1.5$, and $r(k) = 1$ into Equation (1), we obtain

$$\begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ -0.25 & -0.5 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Assume that the initial state is

$$\begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

where a and b are arbitrary. Then

$$\begin{bmatrix} x(1) \\ u(1) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ -0.25 & -0.5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5\,a + b \\ -0.25\,a - 0.5\,b + 1 \end{bmatrix}$$

$$\begin{bmatrix} x(2) \\ x(2) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ -0.25 & -0.5 \end{bmatrix} \begin{bmatrix} 0.5\,a + b \\ -0.25\,a - 0.5\,b + 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

and

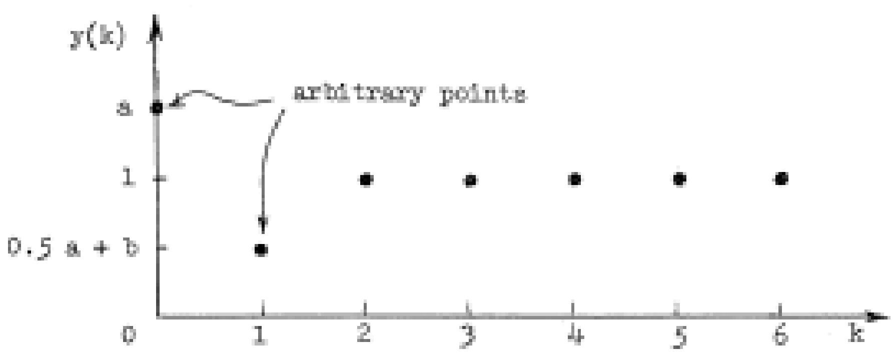
$$\begin{bmatrix} x(k) \\ u(k) \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \qquad \text{for } k = 3, 4, 5, \ldots$$

Hence

$$y(0) = x(0) = a$$

$$y(1) = x(1) = 0.5\,a + b$$

$$y(k) = x(k) = 1 \qquad \text{for } k = 2, 3, 4, \ldots.$$



A sample response plot is shown above.