# V State Space Analysis

## Topics to be covered

1. Introduction
2. State Space Representations of Discrete Time Systems
3. Solving Discrete Time State Space Equations
4. Pulse Transfer Function Matrix
5. Discretization of Continuous-time State Space Equations
6. Liapunov Stability Analysis

# V.1. Introduction

Note:
1) Conventional methods are simple, and only applied to LTI system.
2) State space method can be applied to MIMO system
3) State space method can be used for optimal control

**Definitions:**

**State:** The smallest set of variables called state variables such that the knowledge of these variables at $t = t_0$, together with knowledge of the input for $t > t_0$, completely determines the behavior of the system for any time $t \geq t_0$

**State variables**: The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system.

**State vector**: If $n$ state variables are needed to completely describe the behavior of a given system, these $n$ state variable can be considered the $n$ components of a vector $x$

**State space**: the n-dimensional space whose coordinate axes consist of the $x_1$ axis, $x_2$ axis, $x_n$ axis is called a state space.

**State-space equations**: for a time varying (linear or nonlinear) discrete-time systems, the state equation may be written as :

$$x(k+1) = f[x(k), u(k), k]$$

Output equation is:

$$y(k) = g[x(k), u(k), k]$$

For linear time-varying discrete-time systems, the state equation and output equation may be simplified as:

$$x(k+1) = G(k)x(k) + H(k)u(k)$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

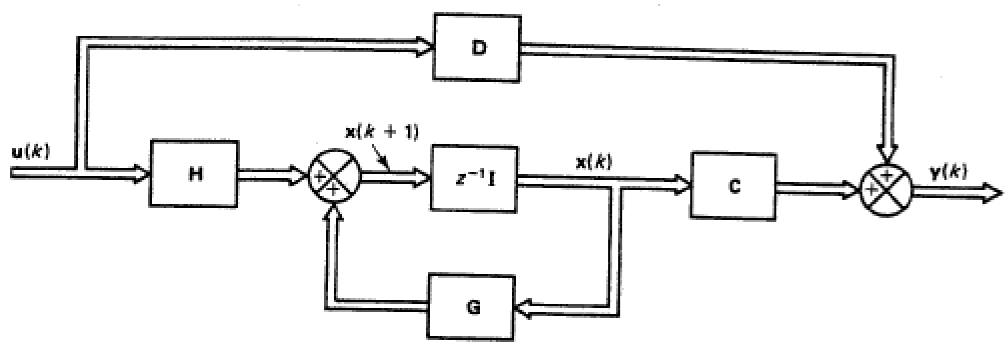
Where

| | |
|---|---|
| $x(k)$=n-vector | (state vector) |
| $y(k)$=m-vector | (output vector) |
| $u(k)$=r-vector | (input vector) |
| $G(k)$=$n \times n$ matrix | (state matrix) |
| $H(k)$=$n \times r$ matrix | (input matrix) |
| $C(k)$=$m \times n$ matrix | (output matrix) |
| $D(k)$=$m \times r$ matrix | (direct transmission matrix) |

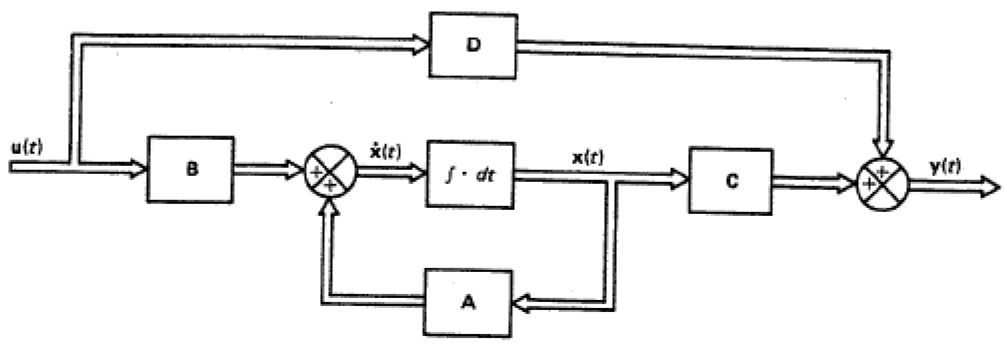
If the system is LTI, the equation can be simplified as

$$x(k+1) = Gx(k) + Hu(k)$$
$$y(k) = Cx(k) + Du(k)$$

Where, $G, H, C, D$ matrix are constant.




(a)




(b)

a) Is the block diagram of the linear time-invariant discrete-time control system representation in state space.
b) Is the block diagram of the linear time-invariant continuous time control system represented in state space.

# V.2. State Space Representations of Discrete Time Systems

Considered a discrete-time system described by

$$y(k)+a_1y(k-1)+a_2y(k-2)+\cdots+a_ny(k-n)=b_0u(k)+b_1u(k-1)+\cdots+b_nu(k-n)$$

Where $u(k)$ is the input and $y(k)$ is the output

Equation can be written in the form of pulse transfer function as

$$\frac{Y(z)}{U(z)}=\frac{b_0+b_1z^{-1}+\cdots+b_nz^{-n}}{1+a_1z^{-1}+\cdots+a_nz^{-n}}$$

Or

$$\frac{Y(z)}{U(z)}=\frac{b_0z^n+b_1z^{n-1}+\cdots+b_n}{z^n+a_1z^{n-1}+\cdots+a_n}$$

Note: there are many ways to realize the state-space representation for the discrete time system.

A) Controllable canonical form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k)=\begin{bmatrix} (b_n-a_nb_0) & (b_{n-1}-a_{n-1}b_0) & \cdots & (b_1-a_1b_0) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + b_0u(k)$$

If we reverse the order of the state variables:

$$\begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \vdots \\ \hat{x}_{n-1}(k) \\ \hat{x}_n(k) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & .\cdot^{\cdot} & \vdots & \vdots \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix}$$

Then the state equation can be written as follows:

$$\begin{bmatrix} \hat{x}_1(k+1) \\ \hat{x}_2(k+1) \\ \vdots \\ \hat{x}_{n-1}(k+1) \\ \hat{x}_n(k+1) \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \vdots \\ \hat{x}_{n-1}(k) \\ \hat{x}_n(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} (b_1 - a_1 b_0) & \cdots & (b_{n-1} - a_{n-1} b_0) & (b_n - a_n b_0) \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \vdots \\ \hat{x}_{n-1}(k) \\ \hat{x}_n(k) \end{bmatrix} + b_0 u(k)$$

B) Observable canonical form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -a_n \\ 1 & 0 & \cdots & 0 & 0 & -a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_2 \\ 0 & 0 & \cdots & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u(k)$$

$$y(k)=\begin{bmatrix}0 & 0 & \cdots & 0 & 1\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\\ \vdots \\ x_{n-1}(k)\\ x_n(k)\end{bmatrix}+b_0u(k)$$

Now if we reverse the order of state variable:

$$\begin{bmatrix}\hat{x}_1(k)\\ \hat{x}_2(k)\\ \vdots \\ \hat{x}_{n-1}(k)\\ \hat{x}_n(k)\end{bmatrix}=\begin{bmatrix}0 & 0 & \cdots & 0 & 1\\ 0 & 0 & \cdots & 1 & 0\\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & 0 & 0 & 0\\ 1 & 0 & \cdots & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\\ \vdots \\ x_{n-1}(k)\\ x_n(k)\end{bmatrix}$$

Then the observable canonical form can be written as follows:

$$\begin{bmatrix}\hat{x}_1(k+1)\\ \hat{x}_2(k+1)\\ \vdots \\ \hat{x}_{n-1}(k+1)\\ \hat{x}_n(k+1)\end{bmatrix}=\begin{bmatrix}-a_1 & 1 & 0 & \cdots & 0 & 0\\ -a_2 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & 0\\ -a_{n-1} & 0 & 0 & \cdots & 0 & 1\\ -a_n & 0 & 0 & \cdots & 0 & 0\end{bmatrix}\begin{bmatrix}\begin{bmatrix}\hat{x}_1(k)\\ \hat{x}_2(k)\\ \vdots \\ \hat{x}_{n-1}(k)\\ \hat{x}_n(k)\end{bmatrix}\end{bmatrix}+\begin{bmatrix}b_1-a_1b_0\\ b_2-a_2b_0\\ \vdots \\ b_{n-1}-a_{n-1}b_0\\ b_n-a_nb_0\end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix}1 & 0 & \cdots & 0 & 0\end{bmatrix}\begin{bmatrix}\begin{bmatrix}\hat{x}_1(k)\\ \hat{x}_2(k)\\ \vdots \\ \hat{x}_{n-1}(k)\\ \hat{x}_n(k)\end{bmatrix}\end{bmatrix}+b_0u(k)$$

C) Diagonal canonical form

If the poles of the pulse transfer function are all distinct, then the state-space representation may be put in the diagonal form as follows:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & p_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & p_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

D) Jordan canonical form

If the poles of the pulse transfer function are multiple of the order m at $z = p_1$ and all other poles are distinct, then the state-space representation may be put in the Jordan canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_m(k+1) \\ x_{m+1}(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & p_1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & p_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & p_{m+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & p_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_m(k) \\ x_{m+1}(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

Example 5.1 (B-5-1) Obtain a state space representation of the following pulse transfer function system in the controllable canonical form, observable canonical form, and diagonal canonical form.

$$\frac{Y(z)}{U(z)} = \frac{z^{-1} + 2z^{-2}}{1 + 4z^{-1} + 3z^{-2}}$$

Controllable canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Observable canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Diagonal canonical form.

$$\frac{Y(z)}{U(z)} = \frac{z^{-1} + 2z^{-2}}{1 + 4z^{-1} + 3z^{-2}} = \frac{z^{-1} + 2z^{-2}}{(1 + z^{-1})(1 + 3z^{-1})} = \frac{\frac{1}{2}z^{-1}}{(1 + z^{-1})} + \frac{\frac{1}{2}z^{-1}}{(1 + 3z^{-1})}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Nonuniqueness of state-space representation

Consider the system defined by

$$x(k+1) = Gx(k) + Hu(k)$$
$$y(k) = Cx(k) + Du(k)$$

Let us define a new state vector

$$x(k) = P\hat{x}(k) \Rightarrow x(k+1) = P\hat{x}(k+1)$$

Then,

$$P\hat{x}(k+1) = GP\hat{x}(k) + Hu(k)$$
$$\Rightarrow \hat{x}(k+1) = P^{-1}GP\hat{x}(k) + P^{-1}Hu(k)$$
$$y(k) = Cx(k) + Du(k) = CP\hat{x}(k) + Du(k)$$

Let us define $P^{-1}GP = \hat{G}$, $P^{-1}H = \hat{H}$, $CP = \hat{C}$ and $D = \hat{D}$

$$\hat{x}(k+1) = \hat{G}\hat{x}(k) + \hat{H}u(k)$$
$$y(k) = \hat{C}\hat{x}(k) + \hat{D}u(k)$$

Now since P can be any nonsingular $n \times n$ matrix, there are infinity many state-space representations for a given system.

# V.3. Solving Discrete time State Space Equations

Solution of the linear time-invariant discrete time state equation:

$$x(k+1) = Gx(k) + Hu(k)$$
$$y(k) = Cx(k) + Du(k)$$

$$x(1) = Gx(0) + Hu(0)$$
$$x(2) = Gx(1) + Hu(1) = G^2 x(0) + GHu(0) + Hu(1)$$
$$x(3) = Gx(2) + Hu(2) = G^3 x(0) + G^2 Hu(0) + GHu(1) + Hu(2)$$
$$\vdots$$

$$x(k) = G^k x(0) + \sum_{j=0}^{k-1} G^{k-j-1} Hu(j), \qquad k = 1,2,3,\cdots$$

$$y(k) = Cx(k) + Du(k) = C\left( G^k x(0) + \sum_{j=0}^{k-1} G^{k-j-1} Hu(j) \right) + Du(k)$$
$$= CG^k x(0) + C\sum_{j=0}^{k-1} G^{k-j-1} Hu(j) + Du(k)$$

State Transition Matrix.

For homogeneous state equation: $x(k+1) = Gx(k) \Rightarrow x(k) = \psi(k)x(0)$
Where $\psi(k)$ is a unique $n \times n$ matrix satisfying the condition

$$\psi(k+1) = G\psi(k), \qquad \psi(0) = I$$

We can see that $\psi(k) = G^k$, which is called the state transition matrix.

Now x and y can be written in terms of $\psi(k)$

$$x(k) = \psi(k)x(0) + \sum_{j=0}^{k-1} \psi(k-j-1)Hu(j), \qquad k = 1,2,3,\cdots$$
$$= \psi(k)x(0) + \sum_{j=0}^{k-1} \psi(j)Hu(k-j-1)$$

$$y(k) = Cx(k) + Du(k) = C\left( G^k x(0) + \sum_{j=0}^{k-1} G^{k-j-1} Hu(j) \right) + Du(k)$$

$$= C\psi(k)x(0) + C\sum_{j=0}^{k-1} \psi(k-j-1)Hu(j) + Du(k)$$

$$= C\psi(k)x(0) + C\sum_{j=0}^{k-1} \psi(j)Hu(k-j-1) + Du(k)$$

Z transform approach to the solution of discrete-time state equations:

$x(k+1) = Gx(k) + Hu(k)$ taking the z transform of both sides

$$zX(z) - zx(0) = GX(z) + HU(z)$$

$$\Rightarrow X(z) = (zI - G)^{-1} zx(0) + (zI - G)^{-1} HU(z)$$

We can conclude:

$G^k = Z^{-1}\left[(zI - G)^{-1} z\right]$ and $\sum_{j=0}^{k-1} G^{k-j-1} Hu(j) = Z^{-1}\left[(zI - G)^{-1} HU(z)\right]$

**Example 5.2** Obtain a state transition matrix for following discrete time state equation.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$(zI - G)^{-1} = \begin{bmatrix} z+1 & 0 \\ 0 & z+3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{z+1} & 0 \\ 0 & \frac{1}{z+3} \end{bmatrix}$$

$$\psi(k) = Z^{-1}\left\{ \begin{bmatrix} \frac{1}{z+1} & 0 \\ 0 & \frac{1}{z+3} \end{bmatrix} z \right\} = Z^{-1}\begin{bmatrix} \frac{z}{z+1} & 0 \\ 0 & \frac{z}{z+3} \end{bmatrix} = Z^{-1}\begin{bmatrix} \frac{1}{1+z^{-1}} & 0 \\ 0 & \frac{1}{1+3z^{-1}} \end{bmatrix} = \begin{bmatrix} (-1)^k & 0 \\ 0 & (-3)^k \end{bmatrix}$$

Method for computing $(zI - G)^{-1}$

$$(zI - G)^{-1} = \frac{adj(zI - G)}{|zI - G|}$$

**Solution of Linear Time-varying Discrete time state equation**

$$x(k+1) = G(k)x(k) + H(k)u(k)$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

**The solution of Linear Time-varying Discrete time state equation may be found easily by recursion**

$$\begin{aligned}
x(h+1) &= G(h)x(h) + H(h)u(h) \\
x(h+2) &= G(h+1)x(h+1) + H(h+1)u(h+1) \\
&= G(h+1)(G(h)x(h) + H(h)u(h)) + H(h+1)u(h+1) \\
&= G(h+1)G(h)x(h) + G(h+1)H(h)u(h) + H(h+1)u(h+1) \\
&\vdots
\end{aligned}$$

Let us define the state transition matrix $\psi(k,h)$, which satisfies

$$\psi(k+1,h) = G(k)\psi(k,h), \quad \psi(h,h) = I \text{ , where } k = h, h+1, h+2, \cdots$$

$$\psi(k,h) = G(k-1)G(k-2)\cdots G(h), \qquad k > h$$

$$x(k) = \psi(k,h)x(h) + \sum_{j=h}^{k-1} \psi(k, j+1)H(j)u(j), \qquad k > h$$

$$y(k) = C(k)\psi(k,h)x(h) + C\sum_{j=h}^{k-1} C(k)\psi(k, j+1)H(j)u(j) + D(k)u(k), \ k > h$$

if $G(k)$ is nonsingular for all k values considered, so that the inverse of $\psi(k,h)$ exists, then the inverse of $\psi(k,h)$, denoted by $\psi(h,k)$, is given as follows:

$$\begin{aligned}
\psi^{-1}(k,h) = \psi(h,k) &= [G(k-1)G(k-2)\cdots G(h)]^{-1} \\
&= G(h)^{-1}G(h+1)^{-1}\cdots G(k-1)^{-1}
\end{aligned}$$

Summary on $\psi(k,h)$

1) $\psi(k,k) = I$
2) $\psi(k,h) = G(k-1)G(k-2)\cdots G(h), \qquad k > h$
3) if $\psi(k,h)$ inverse exists, then $\psi^{-1}(k,h) = \psi(h,k)$
4) if $G(k)$ is nonsingular for all k values considered, then
   $\psi(k,i) = \psi(k, j)\psi(j,i), \qquad \text{for any } i, j, k$
5) if $G(k)$ is singular for any k values considered, then
   $\psi(k,i) = \psi(k, j)\psi(j,i), \qquad \text{for } k > j > i$

# V.4. Pulse Transfer Function Matrix

The state space representation of an nth order linear time invariant discrete-time system with r input and m output can be given by

$$x(k+1) = Gx(k) + Hu(k)$$
$$y(k) = Cx(k) + Du(k)$$

Taking the z transform of both sides

$$zX(z) - zx(0) = GX(z) + HU(z)$$
$$\Rightarrow X(z) = (zI - G)^{-1} zx(0) + (zI - G)^{-1} HU(z)$$

**Note**: to determine the pulse transfer function, $x(0) = 0$
Thus

$$Y(z) = CX(z) + DU(z) = C(zI - G)^{-1} HU(z) + DU(z)$$
$$= \left(C(zI - G)^{-1} H + D\right)U(z) = F(z)U(z)$$
$$\Rightarrow$$
$$F(z) = C(zI - G)^{-1} H + D$$

$F(z)$ is called the pulse transfer function matrix. It is an $m \times r$ matrix

**Similarity transformation:**

Recall:

$$x(k+1) = Gx(k) + Hu(k)$$
$$y(k) = Cx(k) + Du(k)$$

Let us define a new state vector

$$x(k) = P\hat{x}(k) \Rightarrow x(k+1) = P\hat{x}(k+1)$$

Then,

$$P\hat{x}(k+1) = GP\hat{x}(k) + Hu(k)$$
$$\Rightarrow \hat{x}(k+1) = P^{-1}GP\hat{x}(k) + P^{-1}Hu(k)$$
$$y(k) = Cx(k) + Du(k) = CP\hat{x}(k) + Du(k)$$

Let us define $P^{-1}GP = \hat{G}$, $P^{-1}H = \hat{H}$, $CP = \hat{C}$ and $D = \hat{D}$

$$\hat{x}(k+1) = \hat{G}\hat{x}(k) + \hat{H}u(k)$$
$$y(k) = \hat{C}\hat{x}(k) + \hat{D}u(k)$$

$$\begin{aligned}
\hat{F}(z) &= \hat{C}\left(zI - \hat{G}\right)^{-1}\hat{H} + \hat{D} \\
&= CP\left(zI - P^{-1}GP\right)^{-1}P^{-1}H + D \\
&= CP(zP - GP)^{-1}H + D \\
&= C(zI - G)^{-1}H + D \\
&= F(z)
\end{aligned}$$

Remarks: The pulse transfer function matrix is invariant under similarity transformation.

# V.5. Discretization of Continuous-time State Space Equations

Review of solution of continuous time state equation:

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

easy to approve:

$$\frac{d}{dt}\left(e^{At}\right) = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{A^k t^k}{k!}\right) = \sum_{k=0}^{\infty} \frac{d}{dt}\left(\frac{A^k t^k}{k!}\right) = \sum_{k=1}^{\infty} \frac{kA^k t^{k-1}}{k!} = A\sum_{k=1}^{\infty} \frac{A^{k-1} t^{k-1}}{(k-1)!} = A\sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = Ae^{At}$$

$$e^{A(t+s)} = e^{At}e^{As}$$

note: $e^{(A+B)t} = e^{At}e^{Bt}, \ \ if \ \ AB = BA$, $e^{(A+B)t} \neq e^{At}e^{Bt}, \ \ if \ \ AB \neq BA$

$$\dot{x} = Ax + Bu$$

Solution:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

The solution of the state equation starting with the initial state $x(t_0)$ is

$$x(t) = e^{At}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$$

**Discretization of continuous time state equation**:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

in the following analysis, we use the notation $kT$ and $(k+1)T$ instead of $k$ and $k+1$

$\dot{x} = Ax + Bu$ will take the following form:

$$x((k+1)T) = G(T)x(kT) + H(T)u(kT)$$

To determine $G(T)$ and $H(T)$, we use $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

We assume that the input $u(t)$ is sampled and fed to a zero-order hold so that all the components of $u(t)$ are constant over the interval between any two consecutive sampling instants, $u(t) = u(kT), \quad for\ kT \le t \le kT + T$

$$x((k+1)T) = e^{A(k+1)T}x(0) + e^{A(k+1)T}\int_0^{(k+1)T} e^{-A\tau}Bu(\tau)d\tau$$

And

$$x(kT) = e^{AkT}x(0) + e^{AkT}\int_0^{kT} e^{-A\tau}Bu(\tau)d\tau$$

Thus we have

$$\begin{aligned}
x((k+1)T) &= e^{AT}x(kT) + e^{A(k+1)T}\int_{kT}^{(k+1)T} e^{-A\tau}Bu(\tau)d\tau \\
&= e^{AT}x(kT) + e^{A(k+1)T}\int_{kT}^{(k+1)T} e^{-A\tau}Bu(\tau)d\tau \\
&= e^{AT}x(kT) + e^{AT}\int_{kT}^{(k+1)T} e^{AkT}e^{-A\tau}Bu(kT)d\tau \\
&= e^{AT}x(kT) + e^{AT}\left\{\int_{kT}^{(k+1)T} e^{-A(\tau-kT)}d\tau\right\}Bu(kT) \\
&= e^{AT}x(kT) + e^{AT}\left\{\int_0^T e^{-A\lambda}d\lambda\right\}Bu(kT) \\
&= e^{AT}x(kT) + \left\{\int_0^T e^{A\lambda}d\lambda\right\}Bu(kT)
\end{aligned}$$

Now we define: $G(T) = e^{AT}$ and $H(T) = \int_0^T e^{A\lambda}d\lambda B$

$$x((k+1)T) = G(T)x(kT) + H(T)u(kT)$$
$$y(kT) = Cx(kT) + Du(kT)$$

**Note:**

1) Above discrete time state equation is called the zero-order hold equivalent of the continuous time state equation
2) For $T << 1$ $G(T) = e^{AT} \rightarrow e^0 = I$ . Thus as the sampling period T becomes very small, $G(T)$ approaches the identity matrix.

Example 5.3: Obtain the discrete time state and output equations and the pulse transfer function (sampling period T=1 second) of the following continuous time system:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{(s+1)(s+2)}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{(s+1)(s+2)} = \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x((k+1)T) = G(T)x(kT) + H(T)u(kT)$$

$$G(T) = e^{AT} = \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix}$$

$$H(T) = \int_0^T e^{A\lambda} d\lambda B = \int_0^T \begin{bmatrix} e^{-\lambda} & 0 \\ 0 & e^{-2\lambda} \end{bmatrix} d\lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-e^{-T} & 0 \\ 0 & \frac{1-e^{-2T}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-e^{-T} \\ \frac{1-e^{-2T}}{2} \end{bmatrix}$$

$$x((k+1)T) = \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix} x(kT) + \begin{bmatrix} 1-e^{-T} \\ \frac{1-e^{-2T}}{2} \end{bmatrix} u(kT)$$

$$F(z)=C(zI-G)^{-1}H+D$$

$$=\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}z-e^{-T} & 0\\ 0 & z-e^{-2T}\end{bmatrix}^{-1}\begin{bmatrix}1-e^{-T}\\ \dfrac{1-e^{-2T}}{2}\end{bmatrix}$$

$$=\frac{1-e^{-T}}{z-e^{-T}}-\frac{\dfrac{1-e^{-2T}}{2}}{z-e^{-2T}}$$

MATLAB approach:

$$\dot{x}=Ax+Bu\Rightarrow x((k+1)T)=G(T)x(kT)+H(T)u(kT)$$

MATLAB Command: $[G,H]=c2d(A,B,T)$

**Time response between two consecutive sampling instants**

$$\dot{x}=Ax+Bu$$
$$y=Cx+Du$$

The solution with initial state $x(t_0)$

$$x(t)=e^{A(t-t_0)}x(t_0)+\int_{t_0}^{t}e^{A(t-\tau)}Bu(\tau)d\tau$$

To obtain the response of the sampled system at $t=kT+\Delta T$ , where $0<\Delta T<T$ we put $t=kT+\Delta T$ , $t_0=kT$ and $u(\tau)=u(kT)$

$$x(kT+\Delta T)=e^{A\Delta T}x(kT)+\int_{kT}^{kT+\Delta T}e^{AkT+\Delta T}e^{-A\tau}Bu(kT)d\tau$$

$$=e^{A\Delta T}x(kT)+\left\{\int_{0}^{\Delta T}e^{A\lambda}d\lambda\right\}Bu(kT)$$

Now we define: $G(\Delta T)=e^{A\Delta T}$ and $H(\Delta T)=\int_{0}^{\Delta T}e^{A\lambda}d\lambda B$

$$x(kT+\Delta T)=G(\Delta T)x(kT)+H(\Delta T)u(kT)$$

$$y(kT+\Delta T)=Cx(kT+\Delta T)+Du(kT)$$

# V.6. Liapunov Stability Analysis

Note: There are two methods of stability analysis due to Liapunov, called the first method and the second method. The first method consists entirely of procedures in which the explicit forms of the solutions of the differential equations or difference equations are used for analysis. The second method does not require the solution of the differential or difference equation.

The second method is applicable to both linear and nonlinear system, time invariant and time varying system.

**Definitions:**

Positive definiteness of Scalar functions:

$$V(x) > 0, \quad \textit{for } x \neq 0, x \in \Omega$$
$$V(0) = 0$$

Positive definiteness of time-varying functions:

$$\exists \ V(x) > 0, \quad \textit{for } x \neq 0, x \in \Omega$$
$$V(0) = 0$$
$$\textit{and}$$
$$V(x,t) > V(x), \quad \textit{for all } t \geq t_0$$
$$V(0,t) = 0, \quad \textit{for all } t \geq t_0$$

Negative definiteness of Scalar functions:
A scalar function $V(x)$ is negative definite if $-V(x)$is positive definite

Positive semidefiniteness of Scalar functions:
A scalar function $V(x)$ is said to be positive semidefinite if it is positive at all state in the region $\Omega$ except at the origin and at certain other states, where it is zero.

Negative semidefiniteness of Scalar functions:
A scalar function $V(x)$ is negative semidefinite if $-V(x)$is positive semidefinite

Indefiniteness of scalar functions:

A scalar function is said to be indefinite if in the region $\Omega$ it assumes both positive and negative values, no matter how small the region $\Omega$ is.

Example 5.4: Assume $x$ to be a vector, classify following functions

a) $V(x) = x_1^2 + x_3^2 + x_3^2$
b) $V(x) = (x_1 + x_2 + x_3)^2$
c) $V(x) = -x_1^2$
d) $V(x) = -(x_1 - x_2)^2$
e) $V(x) = -x_1 x_2$

**Liapunov Function:** A continuous positive definite functions, has continuous first partial derivatives and the first partial derivatives is negative definite or negative semidefinite.

**System:**

$\dot{x} = f(x,t)$, where x is a state vector.
Denote the solution of system is $\phi(t; x_0, t_0)$, where $x = x_0$ at $t = t_0$

**Equilibrium state:**

At state $x_e$, where $f(x_e, t) = 0, \quad for \quad all \quad t$, $x_e$ is called an equilibrium state of the system

Stability in the sense of Liapunov.

Denote a spherical region of radius $r$ about an equilibrium state $x_e$ as

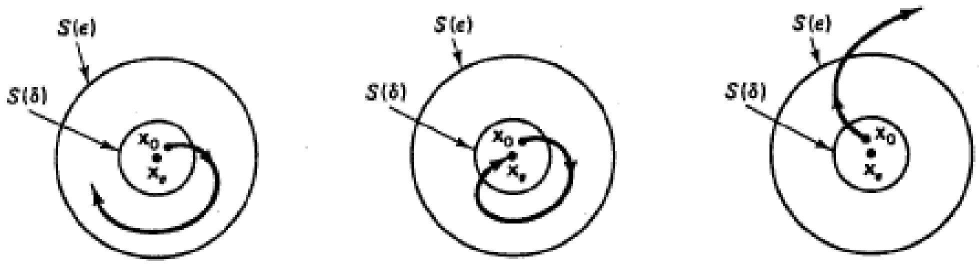
$\|x - x_e\| \le r$, where $\|x - x_e\|$ is called the Euclidean norm and is defined as follows:

$$\|x - x_e\| = \sqrt{\sum_{i=1}^{n} (x_i - x_{ie})^2}$$

Let $S(\delta)$ consist of all points such that $\|x - x_e\| \le \delta$

And let $S(\varepsilon)$ consist of all points such that
$\|\phi(t; x_0, t_0) - x_e\| \le \varepsilon, \quad for\ all\ t \ge t_0$

An equilibrium state $x_e$ of the system $\dot{x} = f(x,t)$ is said to be stable in the sense of Liapunov if, corresponding to each $S(\varepsilon)$, there is an $S(\delta)$ such that trajectories starting in $S(\delta)$ do not leave $S(\varepsilon)$ as t increase indefinitely. $\delta$ is depending on $\varepsilon$ and also $t_0$. If $\delta$ does not depend on $t_0$, then the equilibrium state is said to be uniformly stable.




(a) (b) (c)

Asymptotic stability: An equilibrium state $x_e$ of the system of $\dot{x} = f(x,t)$ is said to be asymptotically stable if it is stable in the sense of Liapunov and if every solution starting within $S(\delta)$ converges, without leaving $S(\varepsilon)$, to $x_e$ as t increases indefinitely.

Asymptotic stability in the large. If asymptotic stability holds for all states from which trajectories originate, the equilibrium state is said to be asymptotically stable in the large.

Instability: An equilibrium state $x_e$ is said to be unstable if for some real number $\varepsilon > 0$ and any real number $\delta > 0$, no matter how small, there is always a state $x_0$ in $S(\delta)$ that the trajectory starting at this state leaves $S(\varepsilon)$.

**Theorem 5-1**

Suppose a system is described by $\dot{x} = f(x,t)$, where $f(0,t) = 0,$ *for all t*
If there exists a scalar function $V(x,t)$ having continuous first partial derivatives and satisfying the conditions

1) $V(x,t)$ is positive definite
2) $\dot{V}(x,t)$ is negative definite

Then the equilibrium state at the origin is uniformly **asymptotically** stable.

If in addition, $V(x,t) \to \infty$ as $\|x\| \to \infty$, then the equilibrium state at the origin is uniformly **asymptotically** stable in the large.

**Theorem 5-2**

Suppose a system is described by $\dot{x} = f(x,t)$, where $f(0,t) = 0,\ \ \textit{for all } t$
If there exists a scalar function $V(x,t)$ having continuous first partial derivatives and satisfying the conditions

1) $V(x,t)$ is positive definite
2) $\dot{V}(x,t)$ is negative semidefinite

Then the equilibrium state at the origin is uniformly stable.

**Theorem 5-3**

Suppose a system is described by $\dot{x} = f(x,t)$, where $f(0,t) = 0,\ \ \textit{for all } t \geq t_0$
If there exists a scalar function $W(x,t)$ having continuous first partial derivatives and satisfying the conditions

1) $W(x,t)$ is positive definite in some region about the origin.
2) $\dot{W}(x,t)$ is positive definite in the same region

Then the equilibrium state at the origin is unstable.

Remarks:

1) In applying Liapunov stability theorems to nonlinear system, the stability conditions obtained from a particular Liapunov function are sufficient conditions but are not necessary conditions.
2) A Liapunov function for a particular system is not unique.

**Theorem 5-4**

Consider the system described by $\dot{x} = Ax$

Where $x$ is a state vector and A is an $n \times n$ constant nonsingular matrix. A necessary and sufficient condition for the equilibrium state $x = 0$ to be asymptotically stable in the large is that, given any positive definite Hermitian matrix Q, there exists a positive definite Hermitian matrix P that $A^*P + PA = -Q$
The scalar function $x^*Px$ is a Lyapunov function for this system.

Remarks: a necessary and sufficient condition for the matrix to be positive definite is that the determinants of all the successive principal minors of the matrix be positive. (If the elements of P are all real, then the Hermitian matrix becomes a real symmetric matrix).

If $\dot{V}(x) = x^* Px$ doesn`t vanish identically along any trajectory, then Q maybe chosen to be positive semi definite.
If an arbitrary positive definite matrix is chosen for Q and the matrix equation $A^* P + PA = -Q$ is solved to determine P, then the positive definiteness of P is necessary and sufficient condition for the asymptotic stability of the equilibrium state $x = 0$
The final result does not depend on the particular Q matrix chosen so long as Q is positive definite.
In determining whether or not there exists a positive definite Hermitian or positive definite real symmetric matrix P, it is convenient to choose Q=I, where I is the identity matrix. Then the elements of P are determined from $A^* P + PA = -I$ and the matrix P is tested for positive definiteness.

Example 5.5: determine the stability of the equilibrium state of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Choose Q=I and substitute it into $A^* P + PA = -I$

$$\begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} p11 & p12 \\ p21 & p22 \end{bmatrix} + \begin{bmatrix} p11 & p12 \\ p21 & p22 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

P12=p21

$$\begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} p11 & p12 \\ p12 & p22 \end{bmatrix} + \begin{bmatrix} p11 & p12 \\ p12 & p22 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

-3p12+-3p12=-1
p12-4p22+p12-4p22=-1
-3p22+p11-4p22=0

We have

$$p12 = \frac{1}{6}$$

$$p22 = \frac{1}{6}$$

$$p11 = \frac{7}{6}$$

since PD, system is asymptotically stable in the large.

$$\Rightarrow p = \begin{bmatrix} \frac{7}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

$$V(x) = x^* Px = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \frac{7}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{7}{6}x_1^2 + \frac{1}{3}x_1 x_2 + \frac{1}{6}x_2^2$$

$$\dot{V}(x) = -x_1^2 - x_2^2$$

Liapunov Stability analysis of discrete-time systems.

Theorem 5-5. Consider the discrete-time system

$$x((k+1)T) = f(x(kT))$$

Where

x is the vector

$f(x)$ is n vector with property that $f(0) = 0$

T sampling period

Suppose there exists a scalar function $V(x)$ continuous in x such that

1. $V(x) > 0 \quad for \quad x \neq 0$
2. $\Delta V(x) < 0 \quad for \quad x \neq 0$

$$\Delta V(x(kT)) = V(x((k+1)T)) - V(x(kT)) = V(f(x(kT))) - V(x(kT))$$

3. $V(0) = 0$
4. $V(x) \to \infty \quad as \quad x \to \infty$

Then the equilibrium state x=0 is asymptotically stable in the large and $V(x)$ is a Liapunov function.

Liapunov Stability Analysis of Linear Time-invariant Discrete-time systems

$$x(k+1) = Gx(k)$$

X is a state vector and G is a constant non singular matrix. The origin x=0 is the equilibrium state.

Chose $V(x(k)) = x^*(k)Px(k)$

$$\begin{aligned} \Delta V(x(k)) &= V(x(k+1)) - V(x(k)) = x^*(k+1)Px(k+1) - x^*(k)Px(k) \\ &= (Gx(k))^* PGx(k) - x^*(k)Px(k) \\ &= x(k)^* G^* PGx(k) - x^*(k)Px(k) \\ &= x(k)^*(G^* PG - P)x(k) \end{aligned}$$

We require $\Delta V(x(k))$ to be negative to ensure asymptotic stability

Therefore : $\Delta V(x(k)) = -x^*(k)Qx(k)$

Where $Q = -(G^* PG - P)$ is PD

**Theorem 5-6** Consider the discrete-time system

$x((k+1)) = Gx(k)$
X is a state vector and G is a constant non singular matrix. The origin x=0 is the equilibrium state.

A necessary and sufficient condition for the equilibrium state $x = 0$ to be asymptotically stable in the large is that, given any positive definite Hermitian matrix Q, there exists a positive definite Hermitian matrix P that $(G^* PG - P) = -Q$
The scalar function $x^* Px$ is a Liapunov function for this system.

If $\Delta Vx((k)) = -x^*(k)Qx(k)$ does not vanish identically along any solution series, then Q maybe chosen to be positive semidefinite.

Stability of a discrete time system obtained by discretizing a continuous time system.

Consider a continuous-time system $\dot{x} = Ax$
And the corresponding discrete-time system $x((k+1)) = Gx(k)$
Where $G = e^{AT}$
If the continuous time system is asymptotically stable, then $\|G^n\| \to 0,\ as\ n \to \infty$