IV Design of Discrete Time Control System by Conventional Methods

Topics to be covered

1. Introduction
2. Mapping between the s plane and z plane
3. Stability analysis
4. Transient and steady state response
5. Design based on root locus method
6. Design based on frequency response method
7. Analytical design method

IV.1. Introduction

Note: Three different design methods are introduced: 1) root locus based method, 2) frequency response method in w plane, 3) analytical method.

Many conventional continuous-time system design method can be applied to discrete-time systems.

IV.2. Mapping between the s plane and z plane

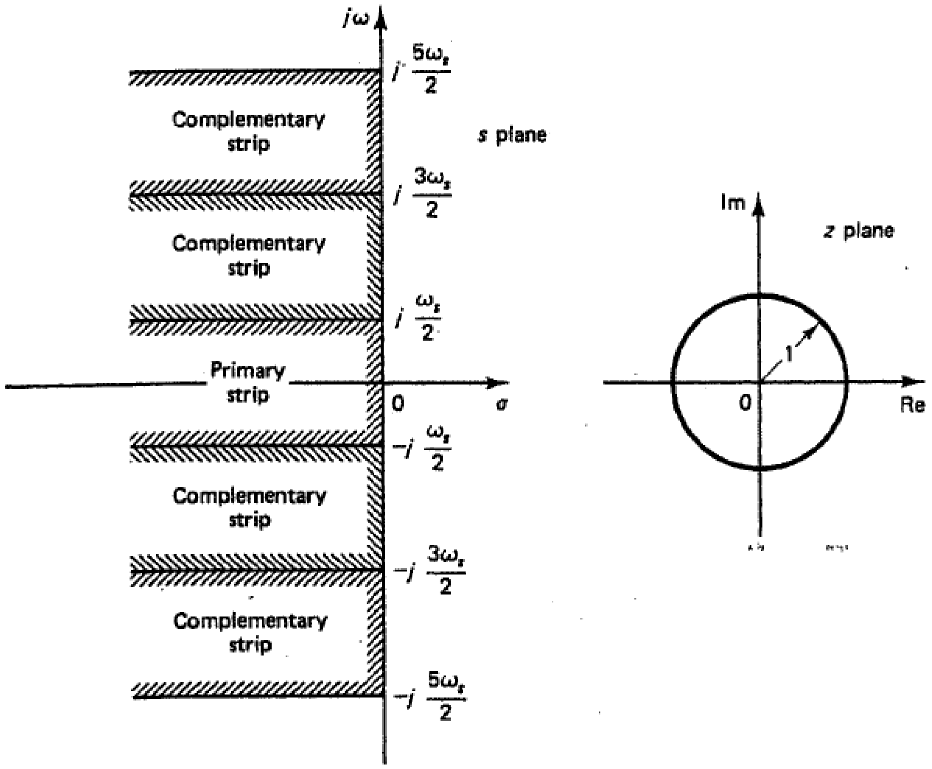
The complex variables z and s are related by the equation

$z = e^{Ts}$

$s = \sigma + jw$

$z = e^{Ts} = e^{T(\sigma + jw)} = e^{T\sigma} e^{jTw} = e^{T\sigma} e^{j(Tw + 2\pi k)}$

Note: frequencies differ in integral multiples of the sampling frequency $\frac{2\pi}{T}$, are mapped into the same location in the z plane. jw axis in the s plane corresponds to $|z| = 1$. The interior of the unit circle corresponds to the left half of the s plane. The exterior of the unit corresponds to the right half of the plane.

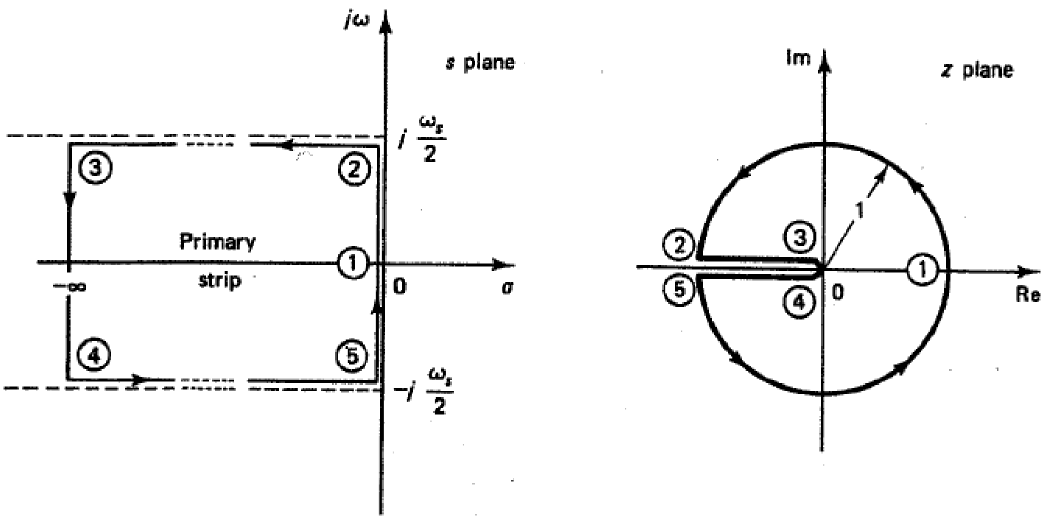


Note:

1) $\angle z = wT$ the angle of z varies from $-\infty \; to \infty$ as w varies from $-\infty \; to \infty$. as point moves from $-j\frac{1}{2}w_s$ to $j\frac{1}{2}w_s$ ($w_s = \frac{2\pi}{T}$) $\angle z = wT$ varies from $-\pi \; to \; \pi$ **This is the primary strip**.

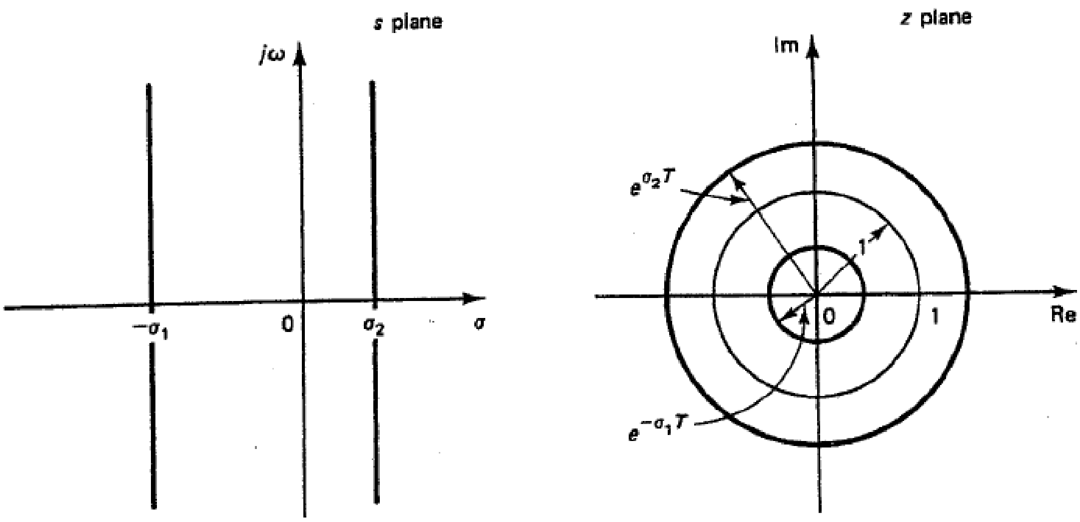
2) Each other strip with range of w_s will trace the z plane in one circle.

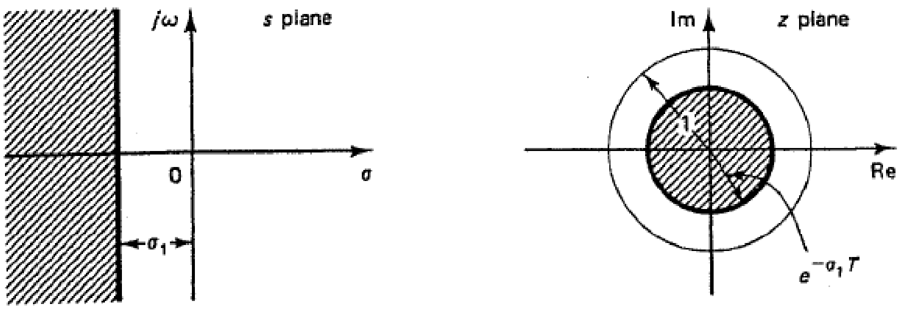
3) Mapping between s plane and z plane is not unique



Mapping between the primary strip and the unit circle

a) constant attenuation line σ is constant maps in to the circle of radius $z = e^{T\sigma}$

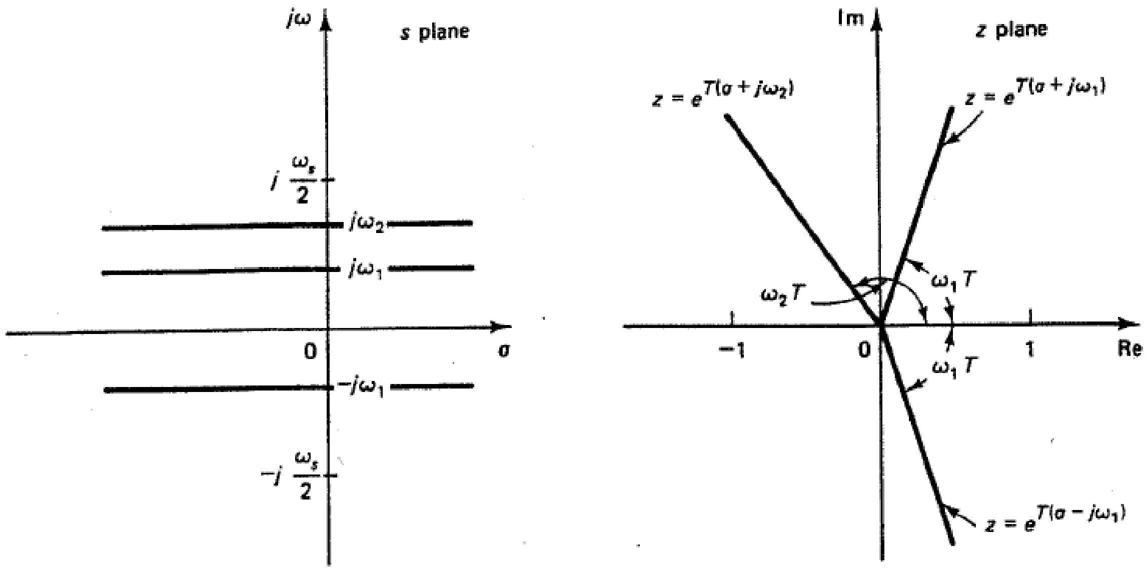




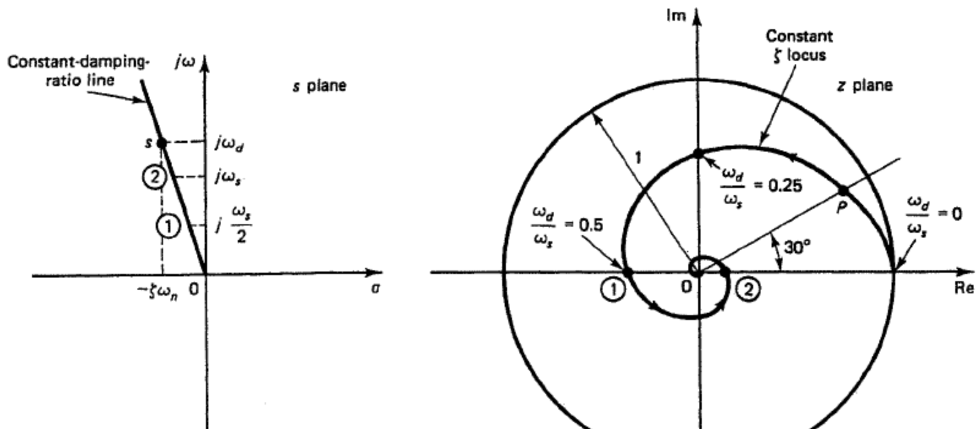
Setting time relate to the region on the left of σ_1, which corresponds to the interior of the circle with radius $z = e^{-T\sigma_1}$

b) Constant frequency loci:

Constant frequency locus $w = w_1$, in the s plane maps into the radial line of constant angle Tw_1 in the z plane.



c) Constant damp ratio line maps into a spiral in the z plane.

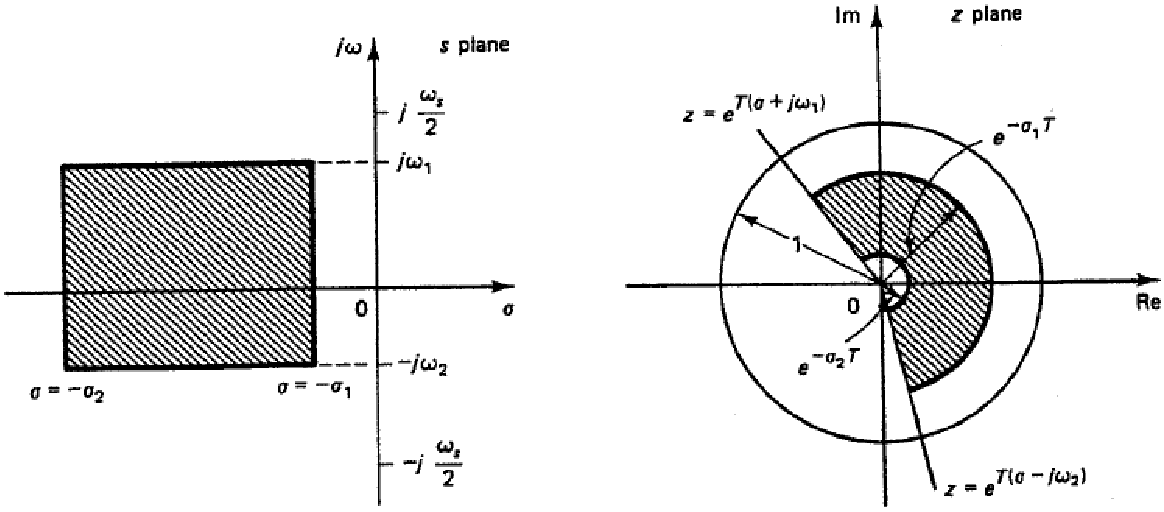


$s = -\xi w_n + j w_n \sqrt{1-\xi^2} = -\xi w_n + j w_d$, where $w_d = w_n \sqrt{1-\xi^2}$

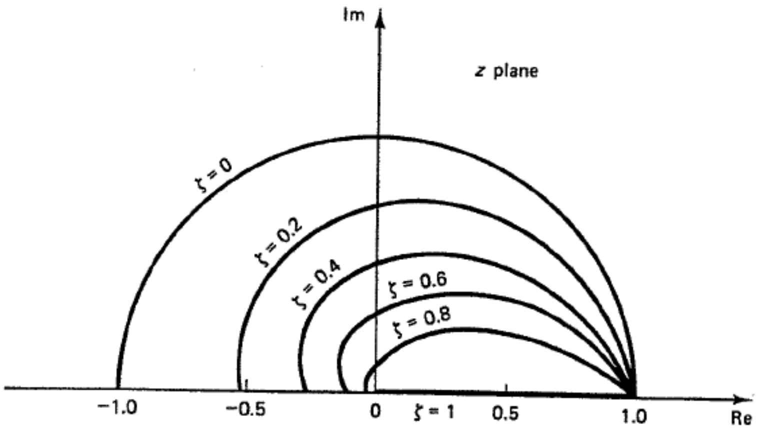
$$z = e^{Ts} = e^{-T\xi w_n + jTw_n\sqrt{1-\xi^2}} = e^{-\frac{2\pi\xi}{\sqrt{1-\xi^2}}\frac{w_d}{w_s} + j2\pi\frac{w_d}{w_s}}$$

Thus $|z| = e^{-\frac{2\pi\xi}{\sqrt{1-\xi^2}}\frac{w_d}{w_s}}$ and $\angle z = 2\pi \dfrac{w_d}{w_s}$

Thus the magnitude of z decreases and the angle of z increases linearly as w_d increases, the locus in the z plane becomes a logarithmic spiral.



The region bounded by constant frequency lines and constant attenuation lines and the mapping.



Constant ξ lines

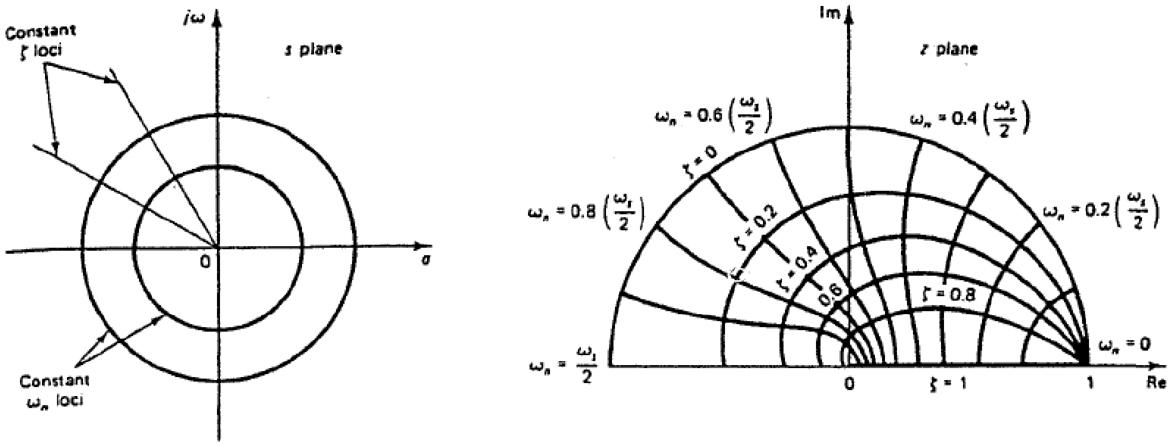
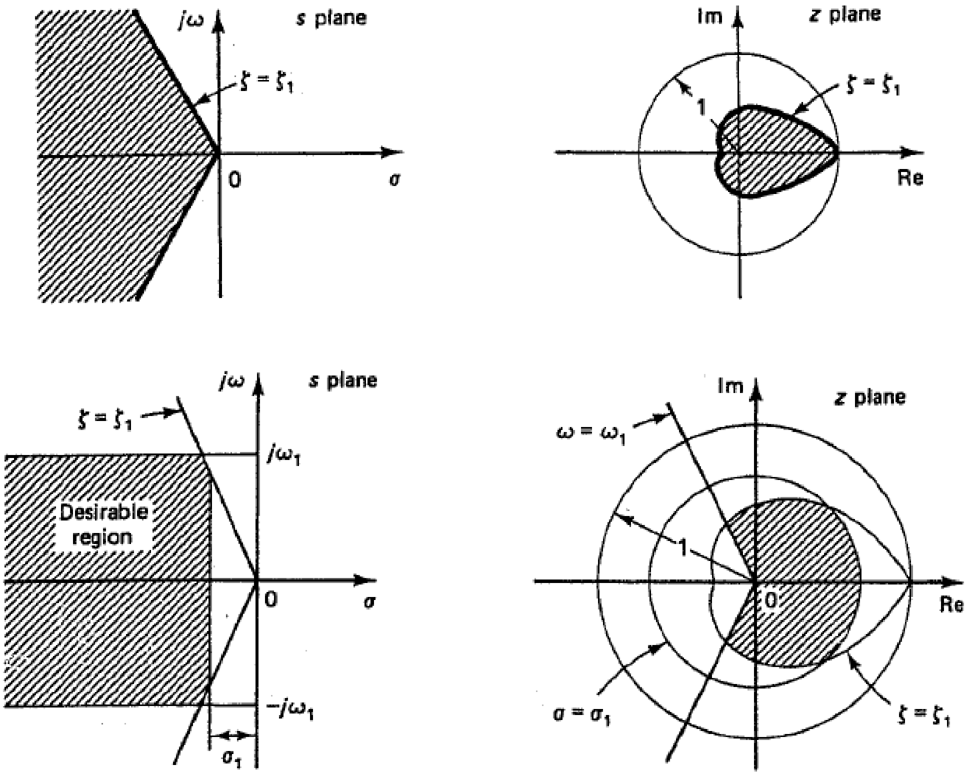


Diagram of the orthogonality of the constant ξ and w_n mapping to the z plane.

Example 4.1.: specify the region in the z plane that corresponds to a desirable region in s plane.



IV.3. Stability analysis

Closed loop pulse transfer function system:

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)} \quad 4.1$$

The stability of 4.2 may be determined from the location of the closed loop poles in the z plane, or the roots of the characteristic equation.

$$P(z) = 1 + GH(z) = 0 \quad 4.2$$

Note:

1) For the stable system, all closed loop poles must lie in the unit circle in the z plane.
2) If a simple pole lies at $z = 1$, then the system becomes critically stable. Also if a single pair of conjugate complex poles lies on the unit circle in the z plane, the system is critically stable. Any multiple closed loop pole on the unit circle makes the system unstable
3) Closed loop zeros do not affect the absolute stability and therefore may be located anywhere in the z plane.

Method for testing absolute stability:

1) Schur-cohn stability test
2) Jury stability test
3) Test based on the bilinear transformation coupled with the Routh stability criterion.
4) Liapunov stability analysis

Note: Both Schur-cohn stability test and Jury stability test may be applied to polynomial equations with real or complex coefficients. When the coefficients are real, Jury test are much simpler than schur-cohn test.

The Jury Stability Test:

$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n$ where $a_0 > 0$

The table is given as follows:

Row	z^0	z^1	z^2	z^3	...	z^{n-2}	z^{n-1}	z^n
1	a_n	a_{n-1}	a_{n-2}	a_{n-3}	...	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3	...	a_{n-2}	a_{n-1}	a_n
3	b_{n-1}	b_{n-2}	b_{n-3}	b_{n-4}	...	b_1	b_0	
4	b_0	b_1	b_2	b_3	...	b_{n-2}	b_{n-1}	
5	c_{n-2}	c_{n-3}	c_{n-4}	c_{n-5}	...	c_0		
6	c_0	c_1	c_2	c_3	...	c_{n-2}		
. . .	. . .							
$2n-5$	p_3	p_2	p_1	p_0				
$2n-4$	p_0	p_1	p_2	p_3				
$2n-3$	q_2	q_1	q_0					

$$b_k = \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix},\ k = 0,1,2,\cdots,n-1$$

$$c_k = \begin{vmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix},\ k = 0,1,2,\cdots,n-2$$

$\vdots$

$$q_k = \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix}\ k = 0,1,2$$

Stability criterion by the Jury Test: A system with the characteristic equation $P(z) = 0$ rewritten as $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n$ where $a_0 > 0$
is stable if the following conditions are all satisfied:

1) $|a_n| < a_0$
2) $P(z)|_{z=1} > 0$
3) $P(z)|_{z=-1} \begin{cases} > 0 & for\ n\ even \\ < 0 & for\ n\ odd \end{cases}$
4) $|b_{n-1}| > |b_0|$
   $|c_{n-2}| > |c_0|$
   $\vdots$
   $|q_2| > |q_0|$

Example 4.2: determine the stability of following Ch equation

$P(z) = z^2 - z + 0.6321 = 0$, we found the roots to be
$z_1 = 0.5 \pm j0.6181 \Rightarrow |z_{1,2}| < 1 \Rightarrow system\ stable$

now use Jury test:

1) $a_0 = 1, a_1 = -1, a_2 = 0.6321 \Rightarrow |a_2| = 0.6321 < a_0 = 1$
2) $P(z)|_{z=1} = 1 - 1 + 0.6321 = 0.6321 > 0$
3) $P(z)|_{z=-1} = 1 + 1 + 0.6321 = 2.6321 > 0 \rightarrow \begin{cases} > 0 & for\ n\ even \\ < 0 & for\ n\ odd \end{cases}$
4) $q_2 = a_0 = 1, q_0 = a_2 = 0.6321 \Rightarrow |q_2| > |q_0|$

All conditions satisfied. the system is stable.

Stability analysis by use of the bilinear transformation and routh stability criterion

Bilinear transformation defined by :

$$z = \frac{w+1}{w-1} \Leftrightarrow w = \frac{z+1}{z-1}$$

bilinear transformation maps the inside of the unit circle into the left half of the w plane.

assume $w = \sigma + jw$

if $|z| < 1 \Rightarrow \left|\frac{\sigma + jw + 1}{\sigma + jw - 1}\right| < 1 \Rightarrow \left|\frac{(\sigma + jw + 1)^2}{(\sigma + jw - 1)^2}\right| < 1 \Rightarrow \frac{(\sigma+1)^2 + w^2}{(\sigma-1)^2 + w^2} < 1 \Rightarrow 4\sigma < 0 \Rightarrow \sigma < 0$

Note: once we transform $P(z) = 0$ into $Q(w) = 0$, it is possible to apply the Routh stability criterion in the same manner.

Example 4.3: determine the stability of following Ch equation using bilinear transformation

$$P(z) = z^2 - z + 0.6321 = 0$$

$$z = \frac{w+1}{w-1}$$

$$\Rightarrow \left(\frac{w+1}{w-1}\right)^2 - \frac{w+1}{w-1} + 0.6321 = 0$$

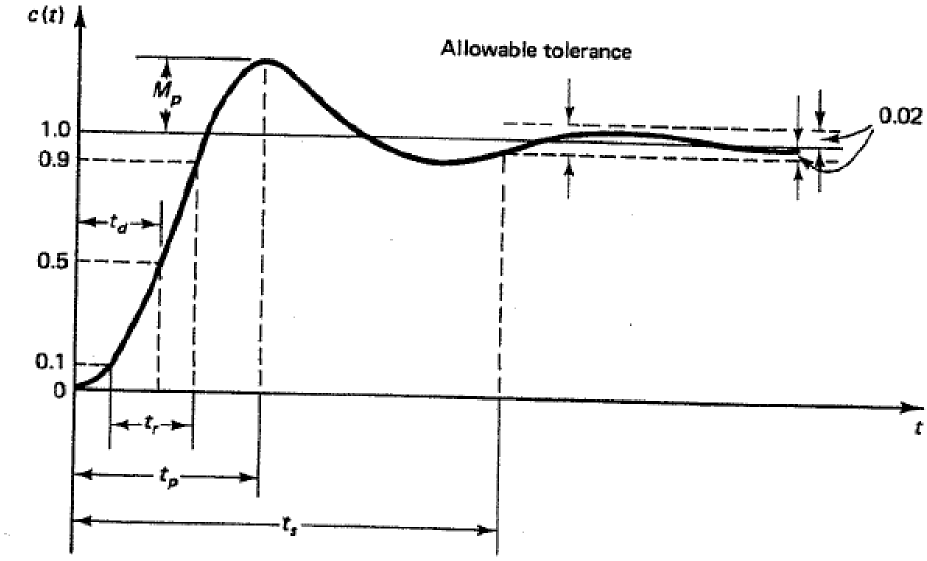
$$\Rightarrow (w+1)^2 - (w+1)(w-1) + 0.6321(w-1)^2 = 0$$

$$\Rightarrow 0.6321w^2 + 0.7358w + 2.6321 = 0$$

s^2	0.6321	2.6321
s^1	0.7358	0
s^0	2.6321	0

so the system is stable.

IV.4. Transient and steady state response



Transient response specifications:

1) Delay time t_d the delay time is the time required for the response to reach half the final value the very first time.
2) Rise time t_r The rise time is the time required for the response to rise from 10% to 90%, or 5% to 95%, or 0% to 100% of its final value.
3) Peak time t_p, the peak time is the time required for the response to reach the first peak of the overshoot.
4) Maximum overshoot M_p The maximum overshoot is the maximum peak value of the response curve measured from unity.

$$\text{maximum percent overshoot} = = \frac{C(t_p) - C(\infty)}{C(\infty)} \times 100\%$$

5) Settling time t_s the settling time is the time required for the response curve to reach and stay within a range about the final value of a size specified as an absolute percentage of the final value, usually 2%.

Note: the transient response of a discrete system to the Kronecker delta input, step input, ramp input, and so on, can be obtained easily by use of MATLAB.

Steady-State Error Analysis:

recall

The loop transfer function is written in general form:

$$G_c(s)G(s) = \frac{k\prod_{i=1}^{M}(s+z_i)}{s^N \prod_{k=1}^{Q}(s+p_k)}$$

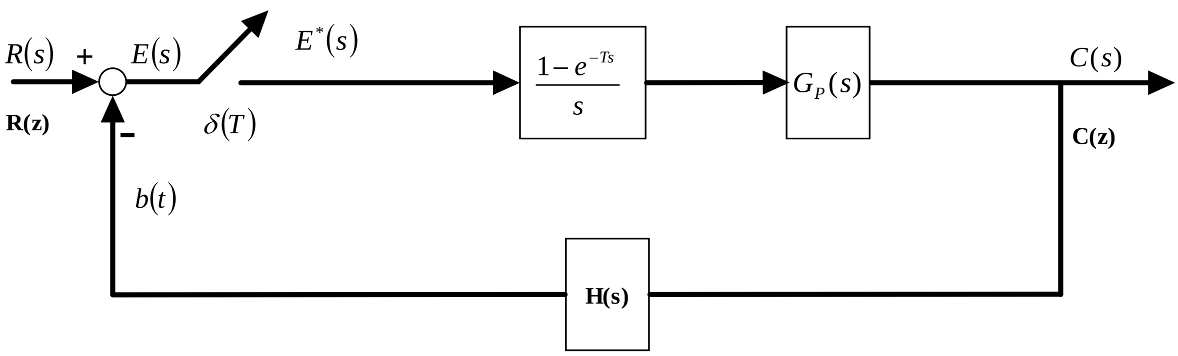
The term s^N in the dominator represents a pole of multiplicity N at the origin. The number of integrations is often indicated by labeling a system with a type number that is simply is equal to N.

Note: as the type number increases, the steady state error reduces, however, stability problem aggravates. Concepts of static error constants can be extended to the discrete-time control system.

The discrete system loop transfer function is written in general form:

$$\frac{C(z)}{R(z)} = \frac{B(z)}{(z-1)^N A(z)} = \frac{k\prod_{i=1}^{M}(z+z_i)}{(z-1)^N \prod_{k=1}^{Q}(z+p_k)}$$

where $\frac{B(z)}{A(z)}$ contains neither a pole or a zero at z=1. Then the system will be classified as type N system.



Consider above system: $e(t) = r(t) - b(t)$

from final value theorem, we have

$$\lim_{k\to\infty} e(kT) = \lim_{z\to 1}(1-z^{-1})E(z) \tag{4.3}$$

for above system configuration :

define

$$G(z) = \left(1 - z^{-1}\right) Z\left[\frac{G_p(s)}{s}\right] \text{ and}$$

$$GH(z) = \left(1 - z^{-1}\right) Z\left[\frac{G_p(s)H(s)}{s}\right]$$

then we have:

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

and

$$E(z) = R(z) - B(z) = R(z) - GH(z)E(z)$$

$$\Rightarrow E(z) = \frac{1}{1 + GH(z)} R(z) \qquad 4.4$$

substitute into 4.3, we have

$$\lim_{k \to \infty} e(kT) = e_{ss} = \lim_{z \to 1} \left(1 - z^{-1}\right) E(z) = \lim_{z \to 1} \left(1 - z^{-1}\right) \frac{1}{1 + GH(z)} R(z)$$

Case 1: Unit step input: $r(t) = u(t) \Rightarrow R(z) = \dfrac{1}{1 - z^{-1}}$

$$e_{ss} = \lim_{z \to 1} \left(1 - z^{-1}\right) E(z) = \lim_{z \to 1} \left(1 - z^{-1}\right) \frac{1}{1 + GH(z)} \frac{1}{\left(1 - z^{-1}\right)} = \lim_{z \to 1} \frac{1}{1 + GH(z)}$$

We define the static position error constant $K_p = \lim_{z \to 1} GH(z)$

thus

$$e_{ss} = \frac{1}{1 + K_p} \qquad 4.5$$

Note: The steady state error in response to a unit step input becomes zero if $K_p = \infty$, which requires that $GH(z)$ has at least one pole at z=1.

Case 2: ramp step input: $r(t) = tu(t) \Rightarrow R(z) = \dfrac{Tz^{-1}}{\left(1 - z^{-1}\right)^2}$

$$e_{ss} = \lim_{z\to 1}\left(1-z^{-1}\right)E(z) = \lim_{z\to 1}\left(1-z^{-1}\right)\frac{1}{1+GH(z)}\frac{Tz^{-1}}{\left(1-z^{-1}\right)^2} = \lim_{z\to 1}\frac{T}{\left(1-z^{-1}\right)GH(z)}$$

We define the static velocity error constant $K_v = \lim_{z\to 1}\frac{\left(1-z^{-1}\right)GH(z)}{T}$

thus

$$e_{ss} = \frac{1}{K_v} \qquad 4.6$$

Note: the steady state error in response to a ram step input becomes zero if $K_v = \infty$, which requires that $GH(z)$ has at least double poles at z=1.

Case 3: Unit acceleration input: $r(t) = \frac{1}{2}t^2u(t) \Rightarrow R(z) = \frac{T^2\left(1+z^{-1}\right)^{-1}z^{-1}}{2\left(1-z^{-1}\right)^3}$

$$e_{ss} = \lim_{z\to 1}\left(1-z^{-1}\right)E(z) = \lim_{z\to 1}\left(1-z^{-1}\right)\frac{1}{1+GH(z)}\frac{T^2\left(1+z^{-1}\right)^{-1}z^{-1}}{2\left(1-z^{-1}\right)^3} = \lim_{z\to 1}\frac{T^2}{\left(1-z^{-1}\right)^2 GH(z)}$$

We define the static acceleration error constant $K_a = \lim_{z\to 1}\frac{\left(1-z^{-1}\right)^2 GH(z)}{T^2}$

thus

$$e_{ss} = \frac{1}{K_a} \qquad 4.7$$

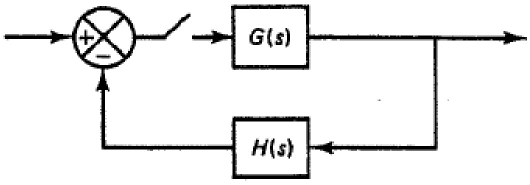
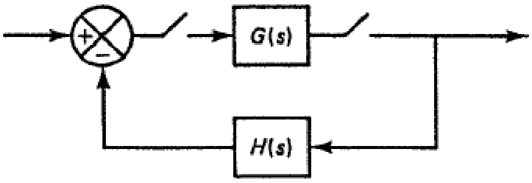
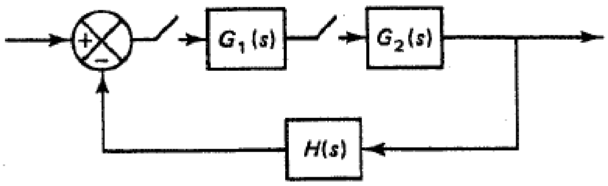
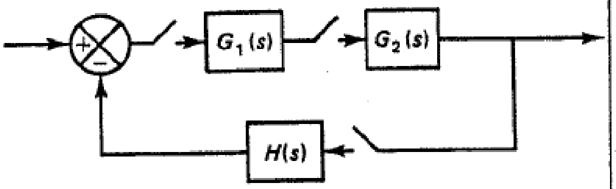
Note: The steady state error in response to a unit acceleration input becomes zero if $K_a = \infty$, which requires that $GH(z)$ has at least three poles at z=1.

Summary:

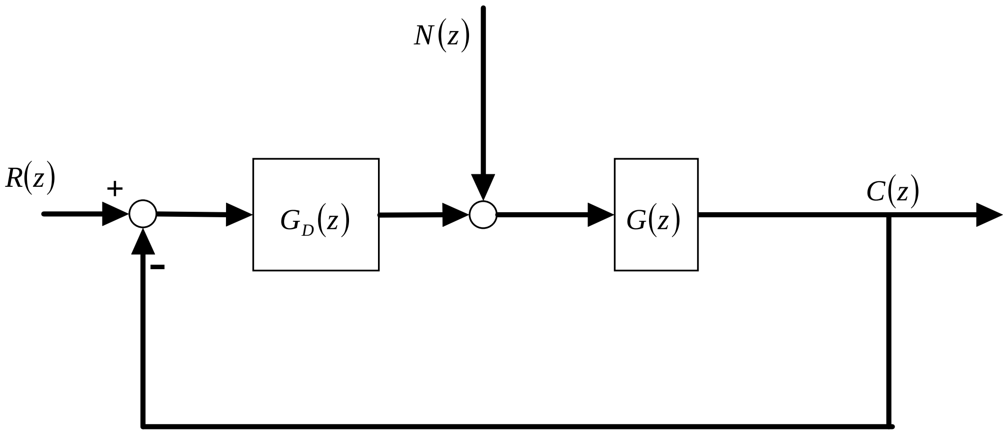
Table of steady state errors

Type number	Step input	Ramp input	Acceleration input
0	$e_{ss} = \frac{A}{1+K_p}$	infinity	infinity
1	0	$e_{ss} = \frac{A}{K_V}$	infinity
2	0	0	$e_{ss} = \frac{A}{K_a}$
3 or more	0	0	0

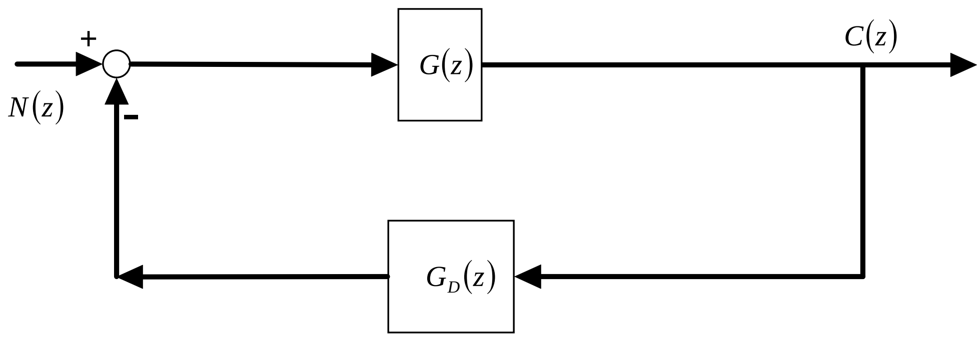
Static error constants for typical closed-loop configurations of discrete time control system

Closed-loop configuration	Values of K_p, K_v, and K_a
	$K_p = \lim_{z \to 1} GH(z)$ $K_v = \lim_{z \to 1} \frac{(1 - z^{-1})GH(z)}{T}$ $K_a = \lim_{z \to 1} \frac{(1 - z^{-1})^2 GH(z)}{T^2}$
	$K_p = \lim_{z \to 1} G(z)H(z)$ $K_v = \lim_{z \to 1} \frac{(1 - z^{-1})G(z)H(z)}{T}$ $K_a = \lim_{z \to 1} \frac{(1 - z^{-1})^2 G(z)H(z)}{T^2}$
	$K_p = \lim_{z \to 1} G_1(z)HG_2(z)$ $K_v = \lim_{z \to 1} \frac{(1 - z^{-1})G_1(z)HG_2(z)}{T}$ $K_a = \lim_{z \to 1} \frac{(1 - z^{-1})^2 G_1(z)HG_2(z)}{T^2}$
	$K_p = \lim_{z \to 1} G_1(z)G_2(z)H(z)$ $K_v = \lim_{z \to 1} \frac{(1 - z^{-1})G_1(z)G_2(z)H(z)}{T}$ $K_a = \lim_{z \to 1} \frac{(1 - z^{-1})^2 G_1(z)G_2(z)H(z)}{T^2}$

Response to disturbance:



(a)



(b)

assume the reference input is zero, or $R(z)=0$

$\frac{C(z)}{N(z)} = \frac{G(z)}{1+G(z)G_D(z)}$, which can be obtained from the equivalent model in b.

$$E(z) = R(z) - C(z) = -C(z) = -\frac{G(z)}{1+G(z)G_D(z)} N(z)$$

$$e_{ss} = \lim_{z\to 1}(1-z^{-1})E(z) = \lim_{z\to 1}(1-z^{-1})\frac{-G(z)}{1+G(z)G_D(z)} N(z)$$

Note: To reduce the disturbance impact, we can increase the gain $|G(z)G_D(z)|$

IV.5. Design based on root locus method

Note: The root locus method developed for continuous-time systems can be extended to discrete-time systems without modifications, except that the stability boundary is changed from the jw axis in the s plane to the unit circle in the z plane.

Angle and magnitude conditions
characteristic equations may have either of following forms:

$1+G(z)H(z)=0$ and $1+GH(z)=0$ which can be written as :

$1+F(z)=0$

Angle condition: $\angle F(z)=\pm 180°(2k+1) \quad k=0,1,2\cdots$
Magnitude condition: $|F(z)|=1$

General procedure for constructing root loci.

1) Obtain the characteristic equation
   $1+F(z)=0$ then rearrange the equation:

$$1+\frac{K(z+z_1)(z+z_2)\cdots(z+z_m)}{(z+p_1)(z+p_2)\cdots(z+p_n)}=0$$

2) Find the starting points and terminating points of the root loci. As K increases from 0 to infinite, a root locus starts from an open loop pole and terminates at a finite open loop zero. Locate the poles p_i and zeros z_i on the z-plane with selected symbols. By conversion, we use ‘x’ to denote poles and ‘o’ to denote zeros.

Remarks:

Loci begins at the poles and ends at the zeros. The number of separate loci is equal to the number of poles.
Root loci must be symmetrical with respect to the horizontal real axis because the complex roots must appear as pairs of complex conjugate roots.

3) Determine the root loci on the real axis. Root loci on the real axis are determined by open-loop poles and zeros lying on it. The root locus on the real axis always lies in a section of the real axis to the left of an odd number of poles and zeros.
4) Determines the asymptotes of the root loci.

The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A. The number of zeros M is less than the number of poles n by N=n-M. the N sections of loci must end at zeros at infinity. These sections of loci proceed to the zeros at infinity along asymptotes as K approaches infinity. These linear asymptotes are centered at a point on the real axis given by

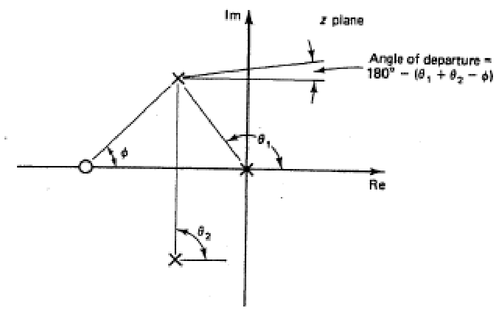
$$\sigma_A = \frac{\sum poles\,of\,P(z) - \sum zeros\,of\,P(z)}{n - M} = \frac{\sum_{i=1}^{n}(-p_i) - \sum_{i=1}^{M}(-z_i)}{n - M}$$

The angle of the asymptotes with respect to the real axis is

$$\phi_A = \frac{2k+1}{n-M}180^0, \text{ k=0,1,2…(n-M-1),}$$

5) Determine the breakaway point on the real axis (if any). In general, the tangent to the loci at the breakaway point are equally spaced over 360^0.
   We may evaluate
   $$\frac{dK}{dz} = 0$$

6) Determine the angle of departure of the locus from a pole and the angle of arrival of the locus at a zero, using the phase angle criterion.
   The angle of locus departure from a pole is the difference between the net angle due to all other poles and zeros and the criterion of $\pm 180^0(2k+1)$, and similarly for the locus angle of arrival at a zero.



7) Find the points where the root loci cross the imaginary axis. set $z = jv$ in the characteristic equation. Equating both the real part and imaginary part to zero. Solving for v and k.
8) Any point on the root loci is a possible closed-loop pole.

Cancellation of the poles of $G(z)$ with zeros of $H(z)$
if $F(z)=G(z)H(z)$ **and the denominator of $G(z)$ and numerator of $H(z)$ have common factors then the corresponding open loop poles and zeros will cancel each other, reducing the degree of the characteristic equation.**

Example 4.4 $H(z)$ **in the feedback loop :** $H(z)=\dfrac{z+c}{z+b}$ **and** $G(z)=\dfrac{z+a}{(z+c)(z+d)}$

$G(z)H(z)=\dfrac{z+c}{z+b}\dfrac{z+a}{(z+c)(z+d)}=\dfrac{z+a}{(z+d)(z+b)}$ **Pole** $(z=-c)$ has been cancelled.

$\dfrac{C(z)}{R(z)}=\dfrac{G(z)}{1+G(z)H(z)}=\dfrac{\dfrac{z+a}{(z+c)(z+d)}}{1+\dfrac{z+a}{(z+b)(z+d)}}=\dfrac{(z+a)(z+b)}{((z+d)(z+b)+z+a)(z+c)}$ **Pole** $(z=-c)$ appears at the closed loop pole.

Example 4.5 $H(z)$ **in the feedforward loop :** $H(z)=\dfrac{z+c}{z+b}$ **and** $G(z)=\dfrac{z+a}{(z+c)(z+d)}$

$G(z)H(z)=\dfrac{z+c}{z+b}\dfrac{z+a}{(z+c)(z+d)}=\dfrac{z+a}{(z+d)(z+b)}$ **Pole** $(z=-c)$ has been cancelled.

$\dfrac{C(z)}{R(z)}=\dfrac{G(z)H(z)}{1+G(z)H(z)}=\dfrac{\dfrac{z+a}{(z+b)(z+d)}}{1+\dfrac{z+a}{(z+b)(z+d)}}=\dfrac{z+a}{((z+d)(z+b)+z+a)}$ **Pole** $(z=-c)$ cancelled at the closed loop pole.

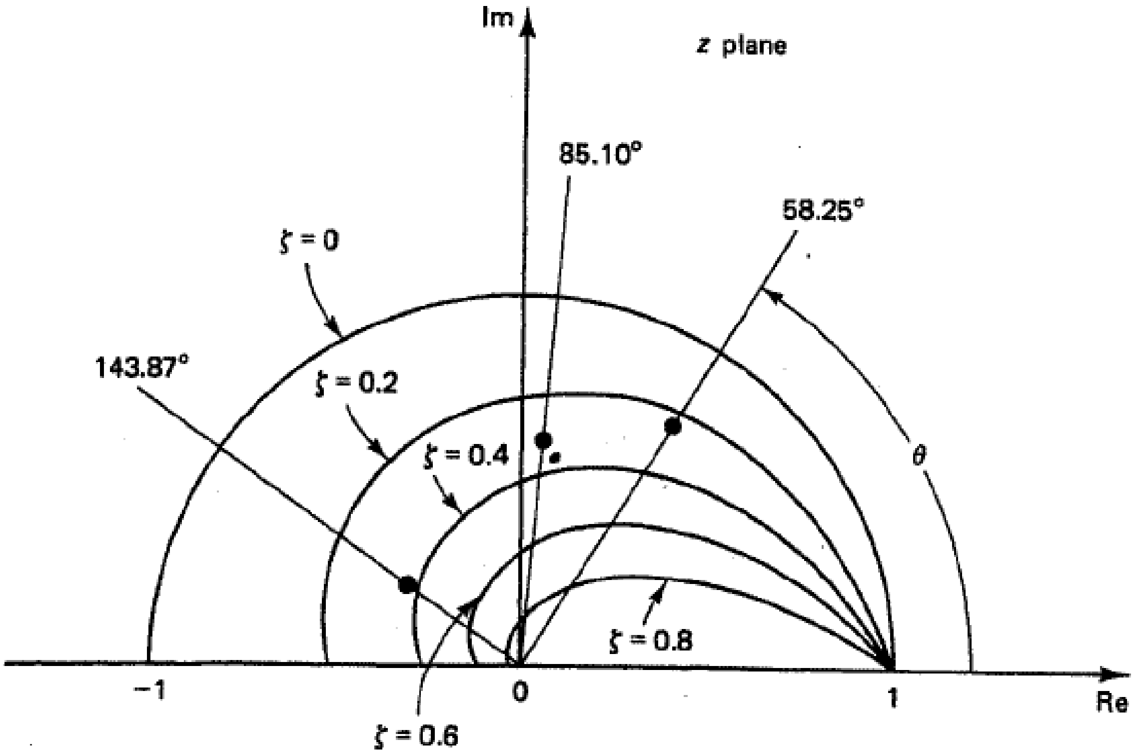
Example 4.6 System has loop gain $G(z)=\dfrac{Tz^{-1}}{(1-z^{-1})}$, draw the root locus.

$1+KG(z)=1+\dfrac{KTz^{-1}}{(1-z^{-1})}=0$

Note: sampling Time will effect the K value. Sometimes, it will effect the pole and zero locations as well.

Effects of sampling Period T on transient response characteristics

Note: Increase the sampling period T will make the system less stable and eventually will make it unstable.



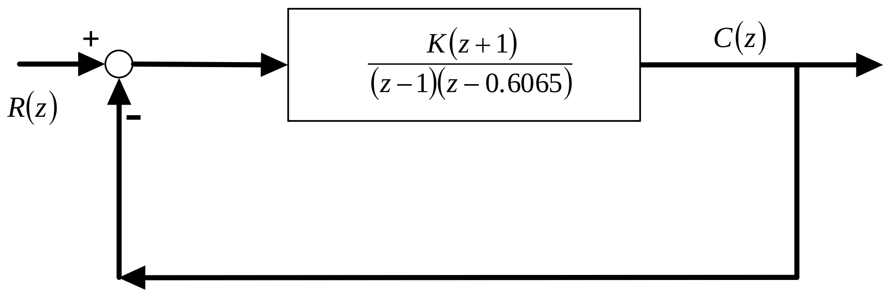
The damp ratio ξ of the closed loop pole can be analytically determined from the location of the closed loop pole in the z plane.

$$s = -\xi w_n + j w_n \sqrt{1-\xi^2} = -\xi w_n + j w_d \text{, where } w_d = w_n \sqrt{1-\xi^2}$$

$$z = e^{Ts} = e^{-T\xi w_n + jTw_n\sqrt{1-\xi^2}} = e^{-\frac{2\pi\xi}{\sqrt{1-\xi^2}}\frac{w_d}{w_s} + j2\pi\frac{w_d}{w_s}}$$

Thus $|z| = e^{-\frac{2\pi\xi}{\sqrt{1-\xi^2}}\frac{w_d}{w_s}}$ and $\angle z = 2\pi\frac{w_d}{w_s}$

Example 4.7 (B-4-8) Consider the digital control system shown in figure, plot the root loci. Determine the critical value of gain K for stability. The sampling period is 0.1 sec what value of gain K will yield a damping ratio ξ of the closed loop poles equal to 0.5? with gain k set to yield $\xi = 0.5$, determine the damped natural frequency w_d and the number of samples per cycle of damped sinusoidal oscillation.



$$G(z) = \frac{K(z+1)}{(z-1)(z-0.6065)}$$

The characteristic equation for the system is

$$z^2 + (K - 1.6065)z + 0.6065 + K = 0$$

The critical value of gain K for stability can be determined easily by use of the Jury stability criterion. Define

$$P(z) = z^2 + (K - 1.6065)z + 0.6065 + K$$
$$= a_0 z^2 + a_1 z + a_2 = 0$$

Then

$$a_0 = 1, \quad a_1 = K - 1.6065, \quad a_2 = 0.6065 + K$$

The conditions for stability are

1. $|a_2| < a_0$
2. $P(1) > 0$
3. $P(-1) > 0$

Thus we require

$$|0.6065 + K| < 1$$
$$P(1) = 1 + K - 1.6065 + 0.6065 + K = 2K > 0$$
$$P(-1) = 1 - K + 1.6065 + 0.6065 + K = 3.213 > 0$$

Hence

$$0 < K < 0.3935$$

The critical value of gain K for stability is 0.3935.

Since

$$G(z) = \frac{K(z+1)}{(z-1)(z-0.6065)}$$

we have

$$\angle G(z) = \angle z+1 - \angle z-1 - \angle z-0.6065$$

Define

$$z = \sigma + j\omega$$

The angle condition is

$$\angle \sigma + j\omega + 1 - \angle \sigma + j\omega - 1 - \angle \sigma + j\omega - 0.6065 = 180^\circ$$

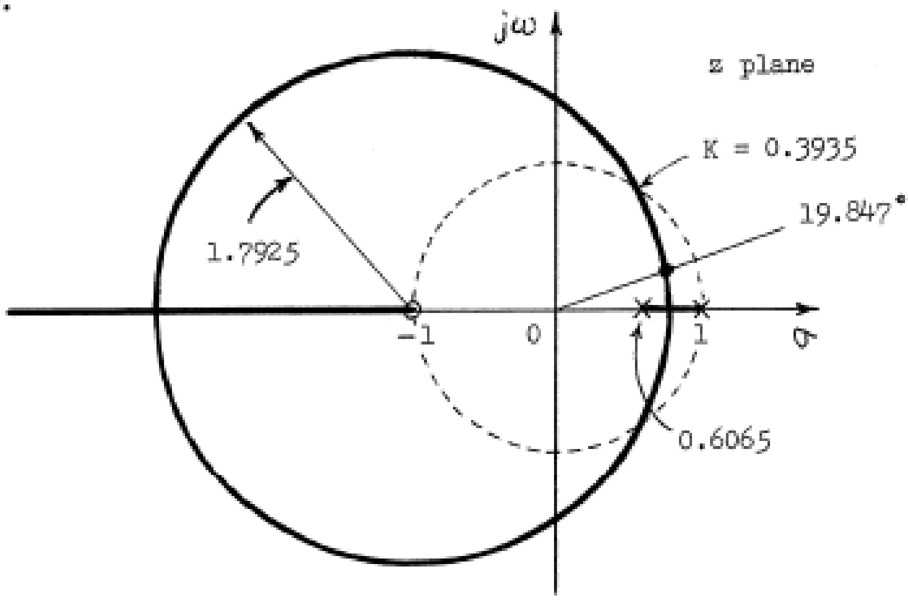
Hence

$$\tan^{-1}\frac{\omega}{\sigma+1} - \tan^{-1}\frac{\omega}{\sigma-1} = 180^\circ + \tan^{-1}\frac{\omega}{\sigma-0.6065}$$

Taking the tangent of both sides of this equation and simplifying, we get

$$\omega = 0 \qquad \text{and} \qquad (\sigma+1)^2 + \omega^2 = (1.7925)^2$$

Thus, the root loci consist of a part of the real axis (between -1 and $-\infty$) and a circle with center at $\sigma = -1$, $\omega = 0$ and the radius equal to 1.7925, as shown below.



The value of gain K that will yield the damping ratio ζ of the closed-loop poles equal to 0.5 can be determined from Equations
Since T is given as 0.1 sec, we have

$$|z| = e^{-0.1 \times 0.5\omega_n} = e^{-0.05\omega_n}$$

$$\angle z = 0.1 \times \omega_n \sqrt{1 - 0.5^2} = 0.0866\ \omega_n$$

By trial and error we find that the point that corresponds to $\zeta = 0.5$ and $\omega_n = 4$ rad/sec, that is, the point for which

$$|z| = e^{-0.05 \times 4} = 0.8187$$

$$\angle z = 0.0866 \times 4 = 0.3464 \text{ rad} = 19.847°$$

is on the root locus. This point is

$$z = e^{-0.05 \times 4} \angle 19.847° = 0.8187 \angle 19.847°$$

$$= 0.7701 + j0.2780$$

The value of gain K that corresponds to this closed-loop pole is found from the magnitude condition

$$\left| \frac{K(z+1)}{(z-1)(z-0.6065)} \right|_{z = 0.7701 + j0.2780} = 1$$

as follows:

$$K = \frac{0.3606 \times 0.3226}{1.7918} = 0.0649$$

When gain K is set to 0.0649, or K = 0.0649, the damping ratio ζ of the dominant closed-loop poles is 0.5. With this gain value, the damped natural frequency ω_d is found as

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 4\sqrt{1 - 0.5^2} = 3.464$$

The number of samples per cycle of the damped sinusoidal oscillation is

$$\frac{360°}{19.847°} = 18.14$$

or 18.14 samples per cycle.

IV.6. Design based on frequency response method

Note: To obtain the frequency response of $G(z)$ we need to only substitute e^{jwT} for z. The function $G(e^{jwT})$ is commonly called the sinusoidal pulse transfer function. It is periodic, with the period equal to T.

Example 4.8 (B-4-8) Consider system defined by $G(z) = \frac{Tz^{-1}}{(1-z^{-1})}$ with input $A \sin kwT$. Obtain the steady state output.

$$G(e^{jwT}) = \frac{T}{(e^{jwT}-1)} = \frac{T}{(\cos(wT) + j\sin(wT) - 1)} = \frac{T}{(\cos(wT) - 1 + j\sin(wT))}$$

$$\left|G(e^{jwT})\right| = \frac{T}{\sqrt{\left((\cos(wT)-1)^2 + (\sin(wT))^2\right)}}$$

$$\angle G(e^{jwT}) = \theta = \tan^{-1}\left(\frac{\sin(wT)}{1-\cos(wT)}\right)$$

Thus the steady state output:

$$x_{ss}(kT) = A\frac{T}{\sqrt{\left((\cos(wT)-1)^2 + (\sin(wT))^2\right)}}\sin\left(kwT + \tan^{-1}\left(\frac{\sin(wT)}{1-\cos(wT)}\right)\right)$$

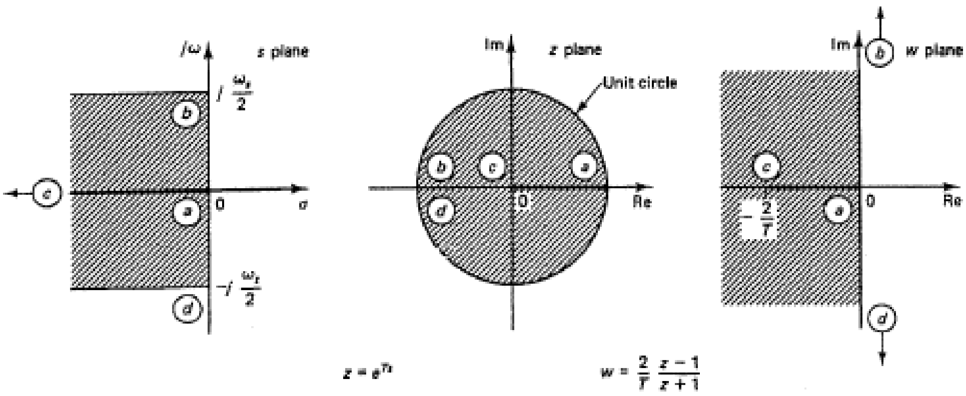
Bilinear transformation and the w plane

Note: z transform maps the primary and complementary strips of the left half of the s plane into the unit circle in the z plane, conventional frequency response methods, which deal with the entire left half plane, do not apply to the z plane.

w transformation will solve the problem:

$$z = \frac{1 + \frac{T}{2} w}{1 - \frac{T}{2} w}$$, where T is the sampling period.

the inverse transform: $w = \frac{2}{T}\frac{z-1}{z+1}$



Note: the primary strip of the left half of the s plane is first mapped into the inside of the unit circle in the z plane and then mapped into the entire left half of the w plane.

s varies from $0 \to j\frac{w_s}{2}$ along jw axis in the s plane

z varies from $1 \to -1$, along the unit circle

w varies from $0 \to \infty$ along the imaginary axis in the w plane.

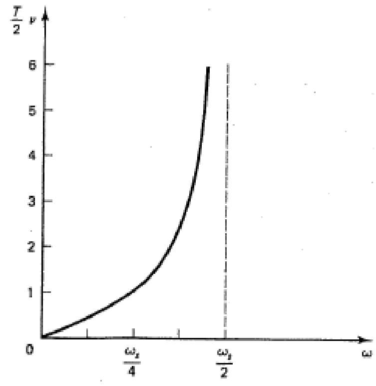
s plane primary strips: $-\frac{w_s}{2} < w < \frac{w_s}{2}$

w plane frequency $0 < v < \infty$

Although w plane resembles the s plane geometrically, the frequency axis in the w plane is distorted. The fictions frequency v and the actual frequency w are related as follows:

$$jv = \frac{2}{T}\frac{z-1}{z+1}\bigg|_{z=e^{jwT}} = \frac{2}{T}\frac{e^{jwT}-1}{e^{jwT}+1} = \frac{2}{T} j \tan\frac{wT}{2} \Rightarrow v = \frac{2}{T}\tan\frac{wT}{2}$$

Following figure shows the relationship between $\frac{T}{2}v$ and w



$G(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n}$, where $m \le n$

take the transformation $z = \frac{1 + \frac{T}{2}w}{1 - \frac{T}{2}w}$

then,

$$G(w) = \frac{\beta_0 w^m + \beta_1 w^{m-1} + \cdots + \beta_n}{\alpha^n + \alpha_1 w^{n-1} + \cdots + \alpha_n}$$

Nyquist stability and bode diagram can be applied.

Note: high frequency asymptote of the logarithmic magnitude for $G(jw)$ and $G(jv)$ may be different.

Advantages of bode diagram approach:

1) low frequency asymptote of the magnitude curve is indicative of the static error constants K_p K_v or K_a
2) Specifications of the transient response can be translated into those of the frequency response in terms of phase margin, gain margin, bandwidth, and so forth.
3) Design in simpler manner.

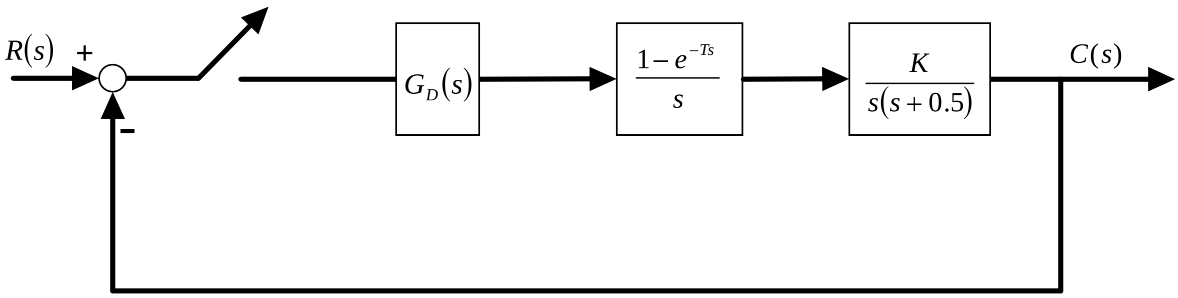
Review phase lead, phase lag, phase lag-lead:

1) Phase lead is commonly used for improving stability margins. The phase lead compensation increases the system bandwidth. Thus the system has faster speed to respond. It may be subjected to high-frequency noise due to its increased high frequency gains.
2) Phase lag compensation reduces the system gain at higher frequencies without reducing the system gain at lower frequencies. The system bandwidth is reduced and the system has slower response. Steady state accuracy can be improved. High frequency noise can be attenuated.
3) Sometime, phase lag compensator is cascaded with a phase lead compensator. Low frequency gain can be increased, the bandwidth can be maintained. PID controller as an example. (PD as lead and PI as lag)

Design Procedure in the w plane.

1) First obtain $G(z)$, then transform $G(z)$ to $G(w)$ through $z=\frac{1+\frac{T}{2}w}{1-\frac{T}{2}w}$, (T should be properly chosen) rule of thumb is to sample at the frequency 10 times that of the bandwidth of the closed loop system)
2) Substitute $w = jv$ and plot the bode diagram for $G(jv)$
3) Read the bode diagram static error constants, the phase margin, the gain margin.
4) Assume the low frequency gain of the controller $G_D(w)$ is unity, determine the system gain by satisfying the requirement for a given static error constant. Using conventional design technique to design $G_D(w)$
5) Transform the $G_D(w)$ to $G_D(z)$ through the bilinear transformation given by $w=\frac{2}{T}\frac{z-1}{z+1}$
6) Realization the pulse transfer function $G_D(z)$ by computational algorithm.

Example 4.8 (B-4-15) Using the bode diagram approach in the w plane, design a digital controller for the system shown in figure. The design specifications are that the phase margin be 50 degree, the gain margin be at least 10 dB, and the static velocity error constant K_v be 20 $\sec^{-1}$. The sampling period is assumed to be 0.1 sec. after the controller is designed, calculate the number of samples per cycle of damped sinusoidal oscillation.



$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} \frac{K}{s(s + 0.5)}\right] = (1 - z^{-1}) \mathcal{Z}\left[\frac{K}{s^2(s + 0.5)}\right]$$

$$= K \frac{0.004918z^{-1} + 0.004836z^{-2}}{(1 - z^{-1})(1 - 0.9512z^{-1})}$$

$$= 0.004918\, K \frac{z + 0.9835}{(z - 1)(1 - 0.9512)}$$

Noting that T = 0.1 sec, we have

$$z = \frac{1 + \frac{1}{2}Tw}{1 - \frac{1}{2}Tw} = \frac{1 + 0.05w}{1 - 0.05w}$$

Hence

$$G(w) = 0.004918 \frac{K\left(\frac{1 + 0.05w}{1 - 0.05w} + 0.9835\right)}{\left(\frac{1 + 0.05w}{1 - 0.05w} - 1\right)\left(\frac{1 + 0.05w}{1 - 0.05w} - 0.9512\right)}$$

$$= \frac{2K(1 - \frac{1}{20} w)(1 + \frac{1}{2404} w)}{w(1 + \frac{1}{0.5002} w)}$$

Assume that the controller $G_D(w)$ has the unity gain at w = 0, or

$$G_D(0) = 1$$

Then, using the requirement that $K_v = 20 \text{ sec}^{-1}$, we determine gain K.

$$K_v = \lim_{w \to 0} wG_D(w)G(w)$$

$$= \lim_{w \to 0} wG_D(w) \frac{2K(1 - \frac{1}{20} w)(1 + \frac{1}{2404} w)}{w(1 + \frac{1}{0.5002} w)} = 2K = 20$$

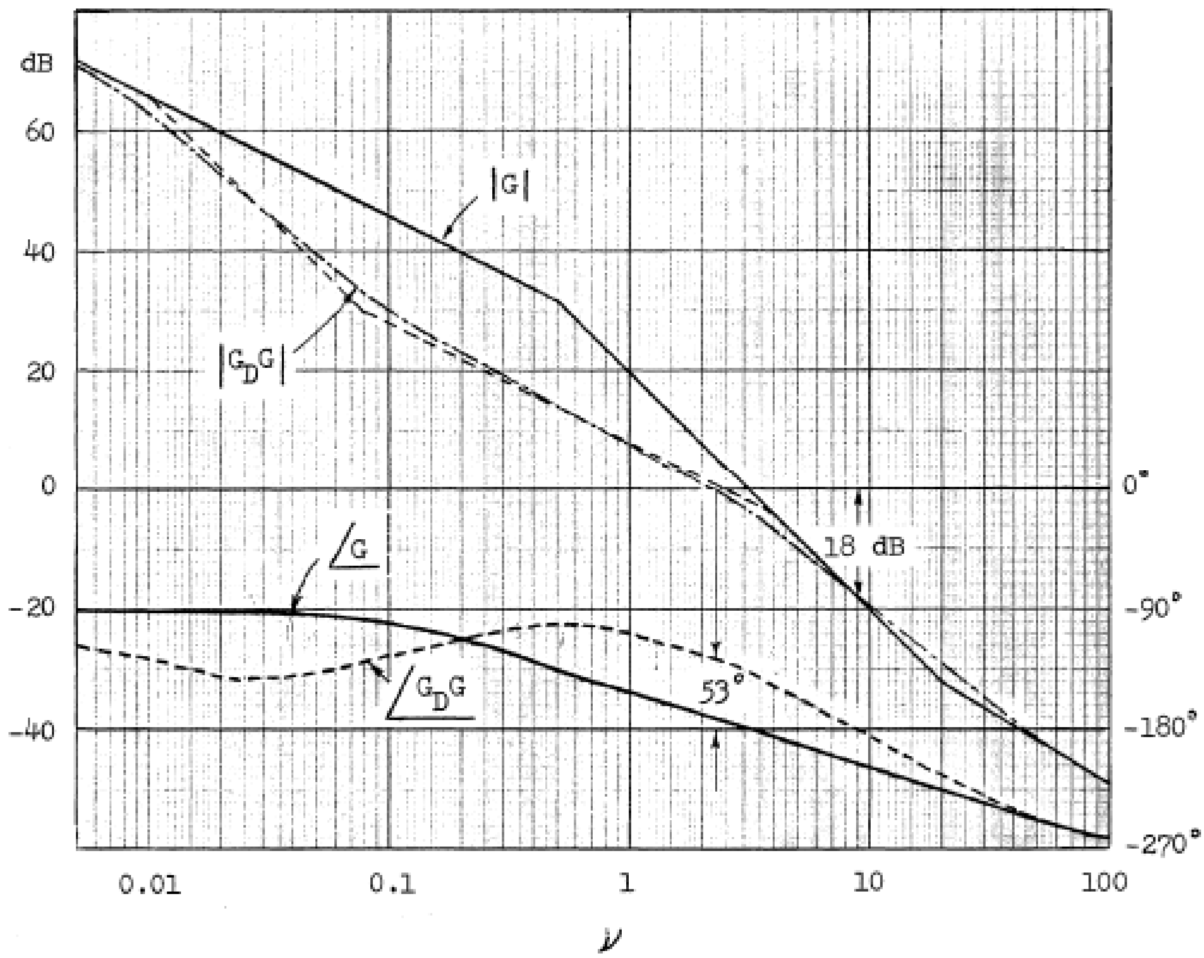
Hence

$$K = 10$$

A Bode diagram of

$$G(w) = \frac{20(1 - \frac{1}{20}w)(1 + \frac{1}{2404}w)}{w(1 + \frac{1}{0.5002}w)}$$

is shown below.



By use of the conventional design technique, we find that the following lag-lead network will satisfy the requirements:

$$G_D(w) = \frac{(1 + \frac{1}{0.08}w)}{(1 + \frac{1}{0.01}w)} \frac{(1 + \frac{1}{0.5}w)}{(1 + \frac{1}{4}w)}$$

The gain crossover frequency is $\nu = 2.3$ rad/sec. The phase margin is approximately 53° and the gain margin is 18 dB.

Next, we transform $G_D(w)$ into $G_D(z)$. Since

$$w = \frac{2}{T}\frac{z - 1}{z + 1} = 20\frac{z - 1}{z + 1}$$

we have

$$G_D(z) = \frac{(1 + \frac{1}{0.08} 20 \frac{z - 1}{z + 1})(1 + \frac{1}{0.5} 20 \frac{z - 1}{z + 1})}{(1 + \frac{1}{0.01} 20 \frac{z - 1}{z + 1})(1 + \frac{1}{4} 20 \frac{z - 1}{z + 1})}$$

$$= 0.8572 \frac{(z - 0.9920)(z - 0.9512)}{(z - 0.9990)(z - 0.6667)}$$

Noting that

$$G(z) = 0.04918 \frac{z + 0.9833}{(z - 1)(z - 0.9512)}$$

we have

$$G_D(z)G(z) = 0.04216 \frac{(z - 0.9920)(z + 0.9833)}{(z - 0.9990)(z - 0.6667)(z - 1)}$$

The characteristic equation of the closed-loop system is

$$(z - 0.9990)(z - 0.6667)(z - 1) + 0.04216(z - 0.9920)(z + 0.9833)$$
$$= 0$$

This is a third degree equation. One root is located near $z = 0.999$. The other two roots are obtained from

$$z^2 - 1.6245z + 0.7080 = 0$$

Thus, the dominant closed-loop poles, which are the roots of this last equation, are located at

$$z = 0.812 \pm j0.220 = 0.841 \angle 15.2^\circ$$

Hence, the number of samples per cycle of damped sinusoidal oscillations is

$$\frac{360^\circ}{15.2^\circ} = 23.7$$

IV.7. Analytical design method

Design of digital controller for minimum settling time with zero steady state error

Define the z transform of the plant that is preceded by the zero-order hold as $G(z)$, or

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s}G_P(s)\right]$$

the open loop transfer function becomes $G_D(z)G(z)$ (feed forward)
Define the closed loop pulse transfer function

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1+G_D(z)G(z)} = F(z) \qquad 4.8$$

Since it is required that the system exhibit a finite settling time with zero steady state error, the system must exhibit a finite impulse response. Hence, $F(z)$ must be of the following form:

$F(z) = a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}$, where $N \geq n$, n is the order of the system.

Note: $F(z)$ must not contain any terms with positive power in z.

solve 4.8 we will have

$$G_D(z) = \frac{F(z)}{G(z)(1-F(z))} \qquad 4.9$$

Note: Both $F(z)$ and $G_D(z)$ must be physically realizable

1) The order of the numerator of $G_D(z)$ must be equal to or lower than the order of the denominator. (Otherwise, the controller requires future input data to generate current input.)
2) If the plant $G_p(s)$ involves a transportation lag e^{-Ls}, then the designed closed loop system must involve at least the same magnitude of the transportation lag. (Otherwise, the close loop system will respond before the input is given)
3) If $G(z)$ is expanded into a series in z^{-1}, the lowest -power term of the series expansion of $F(z)$ in z^{-1} must be at least as large as that of $G(z)$.
4) In addition to the physical realizability conditions, attention should be paid to the stability. We must avoid cancelling an unstable pole of the plant by a zero of the digital controller. Similarly, the digital controller pulse transfer function should not involve unstable poles to cancel plant zeros that lie outside the unit circle.

Assume $G(z)$ involves an unstable pole $G(z)=\frac{G_1(z)}{z-a}$, where $|a|>1$

$$\frac{C(z)}{R(z)}=\frac{G_D(z)G(z)}{1+G_D(z)G(z)}=\frac{G_D(z)\frac{G_1(z)}{z-a}}{1+G_D(z)\frac{G_1(z)}{z-a}}=F(z)$$

, thus $1-F(z)$ must have $z=a$ as zero.

$$\Rightarrow 1-F(z)=\frac{z-a}{z-a+G_1(z)G_D(z)}$$

Note: since $G_D(z)$ should not cancel unstable poles of $G(z)$, all unstable poles must be included in $1-F(z)$ as zeros.
Zeros of $G(z)$ lie inside the unit circle may be cancelled with poles of $G_D(z)$. However, all zeros of $G(z)$ that lie on or outside the unit circle must be included in $F(z)$ as zeros.

Design process:

$$E(z)=R(z)-C(z)=R(z)(1-F(z)) \qquad 4.10$$

for unit step input $R(z)=\frac{1}{1-z^{-1}}$

for unit ramp input $R(z)=\frac{Tz^{-1}}{(1-z^{-1})^2}$

for unit acceleration input: $R(z)=\frac{T^2z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}$

In general, $R(z)=\frac{P(z)}{(1-z^{-1})^{q+1}}$, substitute into 4.10

$$E(z)=R(z)-C(z)=R(z)(1-F(z))$$
$$=\frac{P(z)}{(1-z^{-1})^{q+1}}(1-F(z))$$

to ensure that the system reaches steady state in a finite number of sampling periods and maintains zero steady state error, $E(z)$ must be a polynomial in z^{-1} with a finite number of terms., we chose the function $1-F(z)$ to be the form
$(1-F(z))=(1-z^{-1})^{q+1}N(z)$, where $N(z)$ is a polynomial in z^{-1} with a finite number of terms.
then,
$E(z)=P(z)N(z)$, which is a polynomial in z^{-1} with a finite number of terms.
once we have $F(z)$, then substitute into 4.9 to get $G_D(z)$

$$G_D(z)=\frac{F(z)}{G(z)(1-F(z))}=\frac{F(z)}{G(z)(1-z^{-1})^{q+1}N(z)}$$

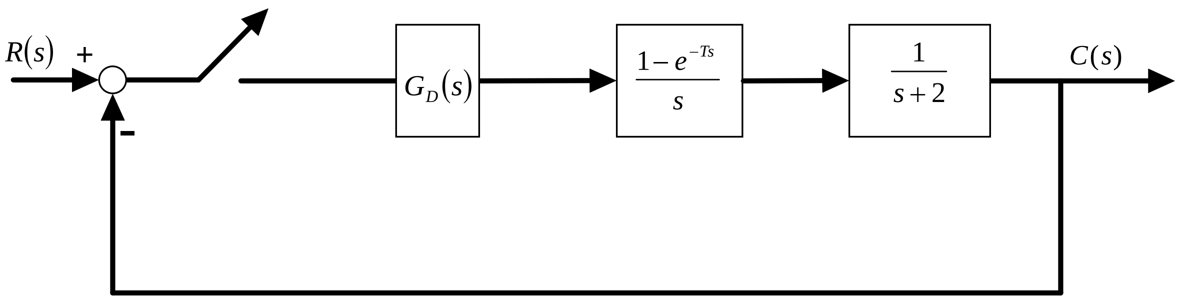
For a stable plant $G_P(s)$, the condition that the output not exhibit intersampling ripples after the settling time is reached my be written as follows:

$C(t \geq nT) = cons\tan t$, for step input

$\dot{C}(t \geq nT) = cons\tan t$, for ramp input

$\ddot{C}(t \geq nT) = cons\tan t$, for acceleration input

Example 4.9 (B-4-18) Consider the control system shown in figure. Design a digital controller $G_D(z)$ such that the system output will exhibit a deadbeat response to a unit step input (that is, the settling time will be the minimum possible and the steady state error will be zero; also the system output will not exhibit intersampling ripples after the settling time is reached) the sampling period T is assumed to be 1 sec.



Since T = 1 sec, we have

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} G_p(s)\right] = (1 - z^{-1})\ \mathcal{Z}\left[\frac{1}{s(s + 2)}\right]$$

$$= \frac{0.4323z^{-1}}{1 - 0.1353z^{-1}}$$

Define the closed-loop pulse transfer function as F(z), or

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)} = F(z) \qquad (1)$$

If G(z) is expanded into a series in z^{-1}, then the first term is $0.4323z^{-1}$. Hence, F(z) must begin with a term in z^{-1}, or

$$F(z) = a_1z^{-1} + a_2z^{-2} + \cdots + a_Nz^{-N} \qquad (2)$$

where $N \geqslant n$ and n is the order of the system.

Since the input is a unit-step function, from Equation (4-48) we have

$$1 - F(z) = (1 - z^{-1})N(z)$$

Notice that G(z) involves neither zero nor pole outside the unit circle. Therefore, there is no requirement on 1 - F(z) from the stability viewpoint.

Since the system should not exhibit intersampling ripples after steady-state is reached, we require U(z) to be of the following type of series in z^{-1}:

$$U(z) = b_0 + b_1z^{-1} + \cdots + b_{N-1}z^{-N+1} + b(z^{-N} + z^{-N-1} + \cdots)$$

Because the plant transfer function $G_p(s)$ does not involve an integrator, b must not be zero. From Figure 4-75,

$$U(z) = \frac{C(z)}{G(z)} = \frac{C(z)}{R(z)}\ \frac{R(z)}{G(z)} = F(z)\ \frac{R(z)}{G(z)}$$

$$= F(z)\ \frac{1}{1 - z^{-1}}\ \frac{1 - 0.1353z^{-1}}{0.4323z^{-1}} \qquad (3)$$

Since U(z) should be of an infinite series, F(z) should not be divisible by $1 - z^{-1}$.

In the absence of other requirements on F(z), we may choose N(z) = 1, or

$$1 - F(z) = 1 - z^{-1}$$

Then

$$F(z) = z^{-1} \tag{4}$$

Thus, in Equation (2), $a_1 = 1$, $a_2 = a_3 = \cdots = a_N = 0$. Clearly, $F(z)$ is not divisible by the factor $1 - z^{-1}$.

From Equation (1) we obtain

$$G_D(z) = \frac{F(z)}{G(z)\left[1 - F(z)\right]} = \frac{z^{-1}}{\dfrac{0.4323z^{-1}}{1 - 0.1353z^{-1}}(1 - z^{-1})}$$

$$= 2.3132 \frac{1 - 0.1353z^{-1}}{1 - z^{-1}}$$

Note that from Equations (3) and (4) we have

$$U(z) = 2.3132 \frac{1 - 0.1353z^{-1}}{1 - z^{-1}}$$

$$= 2.3132 + 2(z^{-1} + z^{-2} + z^{-3} + \cdots)$$

The sequence $u(k)$ in the unit-step response is constant for $k = 1, 2, 3, \ldots$. The system output stays constant at unity and there is no intersampling ripples after the settling time is reached.