ECED 4601 Digital Control Systems

Assignment #6 Reference Solution

http://www.jasongu.org/assignments.html

Assignment #6 contains the following problems:

1) Problem B-7-2: Consider the following Diophantine equation
   $\alpha(z)A(z)+\beta(z)B(z)=1$
   Where
   $A(z)=z^2-0.7z+0.1$
   $B(z)=z^2+0.2z-0.24$
   $\alpha(z)=\alpha_0 z+\alpha_1$
   $\beta(z)=\beta_0 z+\beta_1$

   Solve this Diophantine equation for $\alpha(z)$ and $\beta(z)$

Sylvester matrix $\underset{\sim}{E}$ is given by

$$\underset{\sim}{E} = \begin{bmatrix} 0.1 & 0 & -0.24 & 0 \\ -0.7 & 0.1 & 0.2 & -0.24 \\ 1 & -0.7 & 1 & 0.2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Define matrices $\underset{\sim}{D}$ and $\underset{\sim}{M}$ as follows:

$$\underset{\sim}{D} = \begin{bmatrix} d_3 \\ d_2 \\ d_1 \\ d_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \underset{\sim}{M} = \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{bmatrix}$$

Then $\underset{\sim}{M}$ is obtained from

$$\underset{\sim}{M} = \underset{\sim}{E}^{-1} \underset{\sim}{D}$$

A MATLAB solution for determining $\underset{\sim}{M}$ is shown below.

```
E =

    0.1000         0   -0.2400         0
   -0.7000    0.1000    0.2000   -0.2400
    1.0000   -0.7000    1.0000    0.2000
         0    1.0000         0    1.0000

/ inv(E)

ans =

  -29.5455  -12.2727   -4.6364   -2.0182
  -51.1364  -19.3182   -8.4091   -2.9545
  -16.4773   -5.1136   -1.9318   -0.8409
   51.1364   19.3182    8.4091    3.9545

/ D = [1;0;0;0];
/ M = (inv(E))*D

M =

  -29.5455
  -51.1364
  -16.4773
   51.1364
```

Thus, $\alpha(z)$ and $\beta(z)$ are determined as follows:

$$\alpha(z) = -51.1364z - 29.5455$$

$$\beta(z) = 51.1364z - 16.4773$$

2) B-7-7 consider the plant defined by

$$x((k+1)) = Gx(k) + Hu(k)$$
$$y(k) = Cx(k)$$

Where $G = \begin{bmatrix} 0 & 0 & -0.25 \\ 1 & 0 & 0 \\ 0 & 1 & 0.5 \end{bmatrix}$, $H = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $C = [1 \quad 0 \quad 0]$,

Design a control system for the plant. For the pole placement part, we want to have three closed-loop poles at the origin, or

$H(z) = z^3$

And for the characteristic equation for the minimum order observer, we want to have

$F(z) = z^2$

Use the polynomial equations approach to the design.

First, we determine the transfer function Y(z)/U(z).

$$\frac{Y(z)}{U(z)} = \underset{\sim}{C}(z\underset{\sim}{I} - \underset{\sim}{G})^{-1}\underset{\sim}{H}$$

$$= [1 \quad 0 \quad 0]\begin{bmatrix} z & 0 & 0.25 \\ -1 & z & 0 \\ 0 & -1 & z - 0.5 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{z^2 - 0.75z}{z^3 - 0.5z^2 + 0.25} = \frac{B(z)}{A(z)}$$

Hence

$$A(z) = z^3 - 0.5z^2 + 0.25$$

$$B(z) = z^2 - 0.75z$$

Thus,

$$a_1 = -0.5, \quad a_2 = 0, \quad a_3 = 0.25$$

$$b_0 = 0, \quad b_1 = 1, \quad b_2 = -0.75, \quad b_3 = 0$$

Next, we solve the following Diophantine equation:

$$\alpha(z)A(z) + \beta(z)B(z) = H(z)F(z)$$

or

$$\alpha(z)(z^3 - 0.5z^2 + 0.25) + \beta(z)(z^2 - 0.75z) = z^5$$

where

$$\alpha(z) = \alpha_0 z^2 + \alpha_1 z + \alpha_2$$

$$\beta(z) = \beta_0 z^2 + \beta_1 z + \beta_2$$

To determine $\alpha(z)$ and $\beta(z)$ we define Sylvester matrix $\underset{\sim}{E}$.

$$\underset{\sim}{E} = \begin{bmatrix} 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & -0.75 & 0 & 0 \\ -0.5 & 0 & 0.25 & 1 & -0.75 & 0 \\ 1 & -0.5 & 0 & 0 & 1 & -0.75 \\ 0 & 1 & -0.5 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

We also define matrices $\underset{\sim}{D}$ and $\underset{\sim}{M}$ as follows:

$$\underset{\sim}{D} = \begin{bmatrix} d_5 \\ d_4 \\ d_3 \\ d_2 \\ d_1 \\ d_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \underset{\sim}{M} = \begin{bmatrix} \alpha_2 \\ \alpha_1 \\ \alpha_0 \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix}$$

Then $\underset{\sim}{M}$ is determined from

$$\underset{\sim}{M} = \underset{\sim}{E}^{-1} \underset{\sim}{D}$$

A MATLAB solution for determining $\underset{\sim}{M}$ is shown below.

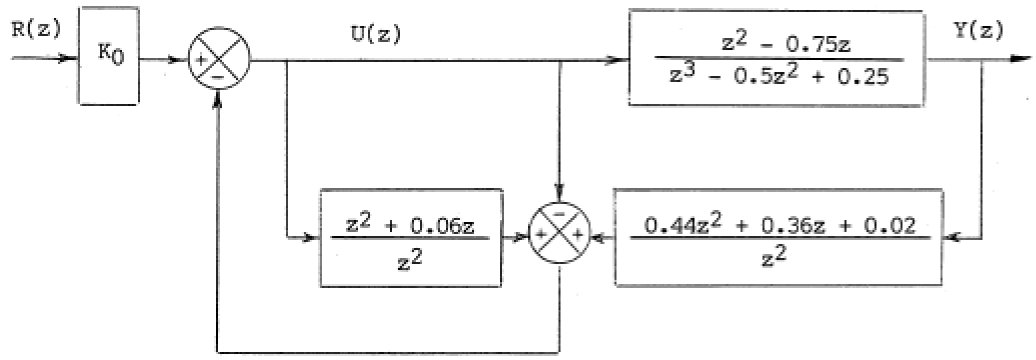
```
E =

    0.2500         0         0         0         0         0
         0    0.2500         0   -0.7500         0         0
   -0.5000         0    0.2500    1.0000   -0.7500         0
    1.0000   -0.5000         0         0    1.0000   -0.7500
         0    1.0000   -0.5000         0         0    1.0000
         0         0    1.0000         0         0         0

/ D = [0;0;0;0;0;1];
/ M = inv(E)*D

M =

         0
    0.0600
    1.0000
    0.0200
    0.3600
    0.4400
```

Thus, $\alpha(z)$ and $\beta(z)$ are determined as follows:

$$\alpha(z) = z^2 + 0.06z$$

$$\beta(z) = 0.44z^2 + 0.36z + 0.02$$

The block diagram for the designed system is shown on next page.



The gain constant K_0 is determined from the requirement that $y(\infty)$ is unity in the unit-step response. Since

$$\frac{Y(z)}{R(z)} = \frac{K_0F(z)B(z)}{H(z)F(z)} = \frac{K_0B(z)}{H(z)} = \frac{K_0(z - 0.75)}{z^2}$$

we have

$$\lim_{k \to \infty} y(k) = \lim_{z \to 1}\left(\frac{z - 1}{z}\right)\left[\frac{K_0(z - 0.75)}{z^2}\right]\left(\frac{z}{z - 1}\right)$$

$$= 0.25K_0 = 1$$

Hence K_0 is determined as

$$K_0 = 4$$

The designed system is of second order.