ECED 4601 Digital Control Systems

Assignment #4 Reference Solution

http://www.jasongu.org/4601/assignments.html

Assignment #4 contains the following problems:

1) Problem B-5-2: obtain a state space representation of the following pulse transfer function system in the observable canonical form.

$$\frac{Y(z)}{U(z)} = \frac{z^{-2} + 4z^{-3}}{1 + 6z^{-1} + 11z^{-2} + 6z^{-3}}$$

Solution:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

2) B-5-15 Obtain the pulse function of the system defined by the equations

$$x(k+1) = Gx(k) + Hu(k)$$
$$y(k) = Cx(k) + Du(k)$$

Where ,

$$G = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

, $C = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & b_3 - a_3 b_0 \end{bmatrix}$ $\qquad D = b_0$

Solution :

Note that

$$(z\underset{\sim}{I} - \underset{\sim}{G})^{-1} = \begin{bmatrix} z + a_1 & a_2 & a_3 \\ -1 & z & 0 \\ 0 & -1 & z \end{bmatrix}^{-1}$$

$$= \frac{1}{z^3 + a_1 z^2 + a_2 z + a_3} \begin{bmatrix} z^2 & -(a_2 z + a_3) & -a_3 z \\ z & z^2 + a_1 z & -a_3 \\ 1 & z + a_1 & z^2 + a_1 z + a_2 \end{bmatrix}$$

Hence

$$(z\underset{\sim}{I} - \underset{\sim}{G})^{-1}\underset{\sim}{H} = \frac{1}{z^3 + a_1 z^2 + a_2 z + a_3} \begin{bmatrix} z^2 \\ z \\ 1 \end{bmatrix}$$

Thus,

$$\underset{\sim}{C}(z\underset{\sim}{I} - \underset{\sim}{G})^{-1}\underset{\sim}{H}$$
$$= \left[b_1 - a_1 b_0 \;\vdots\; b_2 - a_2 b_0 \;\vdots\; b_3 - a_3 b_0 \right] \frac{1}{z^3 + a_1 z^2 + a_2 z + a_3} \begin{bmatrix} z^2 \\ z \\ 1 \end{bmatrix}$$

and

$$F(z) = \underset{\sim}{C}(z\underset{\sim}{I} - \underset{\sim}{G})^{-1}\underset{\sim}{H} + D$$
$$= \frac{(b_1 - a_1 b_0)z^2 + (b_2 - a_2 b_0)z + (b_3 - a_3 b_0)}{z^3 + a_1 z^2 + a_2 z + a_3} + b_0$$
$$= \frac{b_0 z^3 + b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}$$

3) B-5-21 Determine a Liapunov function $V(x)$ for the following system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & -1.2 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

In the Liapunov equation

$$\underset{\sim}{G}^*\underset{\sim}{P}\underset{\sim}{G} - \underset{\sim}{P} = -\underset{\sim}{Q}$$

let us choose $\underset{\sim}{Q}$ to be $\underset{\sim}{I}$. Then

$$\begin{bmatrix} 1 & 0.5 \\ -1.2 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 1 & -1.2 \\ 0.5 & 0 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} p_{12} + 0.25p_{22} & -1.2p_{11} - 1.6p_{12} \\ -1.2p_{11} - 1.6p_{12} & 1.44p_{11} - p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

which yields

$$p_{12} + 0.25p_{22} = -1$$

$$-1.2p_{11} - 1.6p_{12} = 0$$

$$1.44p_{11} - p_{22} = -1$$

Solving these three equations for p_{11}, p_{12}, and p_{22}, we obtain

$$p_{11} = \frac{10}{3.12}, \quad p_{12} = -\frac{5}{2.08}, \quad p_{22} = \frac{17.52}{3.12}$$

Hence, matrix $\underset{\sim}{P}$ is given by

$$\underset{\sim}{P} = \begin{bmatrix} \frac{10}{3.12} & -\frac{5}{2.08} \\ -\frac{5}{2.08} & \frac{17.52}{3.12} \end{bmatrix}$$

Clearly, matrix $\underset{\sim}{P}$ is positive definite.

A Liapunov function $V(\underset{\sim}{x}(k))$ is given by

$$V(\underset{\sim}{x}(k)) = \underset{\sim}{x}^*(k)\underset{\sim}{P}\underset{\sim}{x}(k)$$