ECED 4601 Digital Control Systems

Assignment #1 Reference Solution

http://www.jasongu.org/4601/assignments.html

Assignment #1 contains the following problems:

1) Problem B-2-2: Obtain the z transform of k^3

Method 1: Noting that

$$\mathcal{Z}[k] = \frac{z^{-1}}{(1 - z^{-1})^2}, \qquad \mathcal{Z}[k^2] = \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$$

it can be expected that $\mathcal{Z}[k^3]$ will involve a term $(1 - z^{-1})^4$ in the denominator. Since

$$\mathcal{Z}[k^3] = \sum_{k=0}^{\infty} k^3 z^{-k} = z^{-1} + 2^3 z^{-2} + 3^3 z^{-3} + 4^3 z^{-4} + \cdots$$

$$= z^{-1} + 8z^{-2} + 27z^{-3} + 64z^{-4} + \cdots$$

and

$$(z^{-1} + 8z^{-2} + 27z^{-3} + 64z^{-4} + \cdots)(1 - z^{-1})^4$$

$$= (z^{-1} + 7z^{-2} + 19z^{-3} + 37z^{-4} + \cdots)(1 - z^{-1})^3$$

$$= (z^{-1} + 6z^{-2} + 12z^{-3} + 18z^{-4} + \cdots)(1 - z^{-1})^2$$

$$= (z^{-1} + 5z^{-2} + 6z^{-3} + 6z^{-4} + \cdots)(1 - z^{-1})$$

$$= z^{-1} + 4z^{-2} + z^{-3}$$

we find

$$\mathcal{Z}[k^3] = \frac{z^{-1} + 4z^{-2} + z^{-3}}{(1 - z^{-1})^4} = \frac{z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}$$

Method 2:

$$\mathcal{Z}[k^3] = \mathcal{Z}[k \cdot k^2] = -z\frac{dX(z)}{dz}$$

where

$$X(z) = \mathcal{Z}[k^2] = \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$$

Since

$$\frac{dX(z)}{dz} = \frac{d}{dz}\left[\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}\right] = -\frac{z^{-2} + 4z^{-3} + z^{-4}}{(1 - z^{-1})^4}$$

We have

$$\mathcal{Z}[k^3] = \frac{z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}$$

2) B2-2-5: find the z transform of

$$x(k) = \sum_{h=0}^{k} a^h$$, where a is a constant

Referring to Problem A-2-4, we have

$$\mathcal{Z}\left[\sum_{h=0}^{k} a^h\right] = \frac{1}{1 - z^{-1}} X(z)$$

where

$$X(z) = \mathcal{Z}\left[a^h\right] = \frac{1}{1 - az^{-1}}$$

Hence

$$\mathcal{Z}\left[\sum_{h=0}^{k} a^h\right] = \frac{1}{1 - z^{-1}} \cdot \frac{1}{1 - az^{-1}}$$

3) B-2-8: obtain the inverse z transform of

$$x(z) = \frac{1 + 2z + 3z^2 + 4z^3 + 5z^4}{z^4}$$

By dividing both numerator and denominator by z^4, we have

$$X(z) = 5 + 4z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

This last equation is already in the form of a power series in z^{-1}. By inspection, we have

$$x(0) = 5$$
$$x(1) = 4$$
$$x(2) = 3$$
$$x(3) = 2$$
$$x(4) = 1$$
$$x(k) = 0 \qquad k \geqslant 5$$

Note that the given X(z) is the z transform of a signal of finite length.

4) B-2-13: by using the inversion method, obtain the inverse z transform of

$$x(z) = \frac{1 + 6z^{-2} + z^{-3}}{\left(1 - z^{-1}\right)\left(1 - 0.2z^{-1}\right)}$$

$$X(z) = \frac{1 + 6z^{-2} + z^{-3}}{(1 - z^{-1})(1 - 0.2z^{-1})} = \frac{z^3 + 6z + 1}{z(z - 1)(z - 0.2)}$$

$$X(z)z^{k-1} = \frac{(z^3 + 6z + 1)z^{k-1}}{z(z - 1)(z - 0.2)}$$

For k = 0:

$$X(z)z^{k-1} = \frac{z^3 + 6z + 1}{z^2(z - 1)(z - 0.2)}$$

Hence

$$x(0) = \left[\text{residue of } \frac{z^3 + 6z + 1}{z^2(z - 1)(z - 0.2)} \text{ at double pole } z = 0\right]$$
$$+ \left[\text{residue of } \frac{z^3 + 6z + 1}{z^2(z - 1)(z - 0.2)} \text{ at pole } z = 1\right]$$
$$+ \left[\text{residue of } \frac{z^3 + 6z + 1}{z^2(z - 1)(z - 0.2)} \text{ at pole } z = 0.2\right]$$
$$= \frac{1}{(2 - 1)!} \lim_{z \to 0} \frac{d}{dz}\left[\frac{z^3 + 6z + 1}{(z - 1)(z - 0.2)}\right]$$
$$+ \lim_{z \to 1}\left[\frac{z^3 + 6z + 1}{z^2(z - 0.2)}\right] + \lim_{z \to 0.2}\left[\frac{z^3 + 6z + 1}{z^2(z - 1)}\right]$$
$$= 60 + 10 - 69 = 1$$

For k = 1:

$$X(z)z^{k-1} = \frac{z^3 + 6z + 1}{z(z - 1)(z - 0.2)}$$

Hence

$$x(1) = \left[\text{residue of } \frac{z^3 + 6z + 1}{z(z - 1)(z - 0.2)} \text{ at pole } z = 0\right]$$
$$+ \left[\text{residue of } \frac{z^3 + 6z + 1}{z(z - 1)(z - 0.2)} \text{ at pole } z = 1\right]$$
$$+ \left[\text{residue of } \frac{z^3 + 6z + 1}{z(z - 1)(z - 0.2)} \text{ at pole } z = 0.2\right]$$
$$= \lim_{z \to 0}\left[\frac{z^3 + 6z + 1}{(z - 1)(z - 0.2)}\right] + \lim_{z \to 1}\left[\frac{z^3 + 6z + 1}{z(z - 0.2)}\right]$$
$$+ \lim_{z \to 0.2}\left[\frac{z^3 + 6z + 1}{z(z - 1)}\right] = 5 + 10 - 13.8 = 1.2$$

For k = 2, 3, 4, ...:

$$X(z)z^{k-1} = \frac{(z^3 + 6z + 1)z^{k-2}}{(z - 1)(z - 0.2)}$$

$$x(k) = \left[\text{residue of } \frac{(z^3 + 6z + 1)z^{k-2}}{(z - 1)(z - 0.2)} \text{ at pole } z = 1\right]$$
$$+ \left[\text{residue of } \frac{(z^3 + 6z + 1)z^{k-2}}{(z - 1)(z - 0.2)} \text{ at pole } z = 0.2\right]$$

$$= \lim_{z \to 1}\left[\frac{(z^3 + 6z + 1)z^{k-2}}{z - 0.2}\right] + \lim_{z \to 0.2}\left[\frac{(z^3 + 6z + 1)z^{k-2}}{z - 1}\right]$$
$$= 10 - 2.76(0.2)^{k-2}$$

In summarizing, we have

$$x(0) = 1$$
$$x(1) = 1.2$$
$$x(k) = 10 - 2.76(0.2)^{k-2} \qquad \text{for } k = 2, 3, 4, \ldots$$

5) B-2-17

Solve the following difference equation:

$$x(k+2) - x(k+1) + 0.25x(k) = u(k+2)$$

Where $x(0) = 1$ and $x(1) = 2$. The input function $u(k) = 1, \quad k = 0,1,2,\cdots$

Solve this problem both analytically and computationally with MATLAB.

$$x(k+2) - x(k+1) + 0.25x(k) = u(k+2)$$

The z transform of this difference equation is

$$\left[z^2X(z) - z^2x(0) - zx(1)\right] - \left[zX(z) - zx(0)\right] + 0.25X(z) = z^2U(z) - z^2u(0) - zu(1)$$

Substituting the initial data into this last equation, we get

$$(z^2 - z + 0.25)X(z) = \frac{z^3}{z-1}$$

or

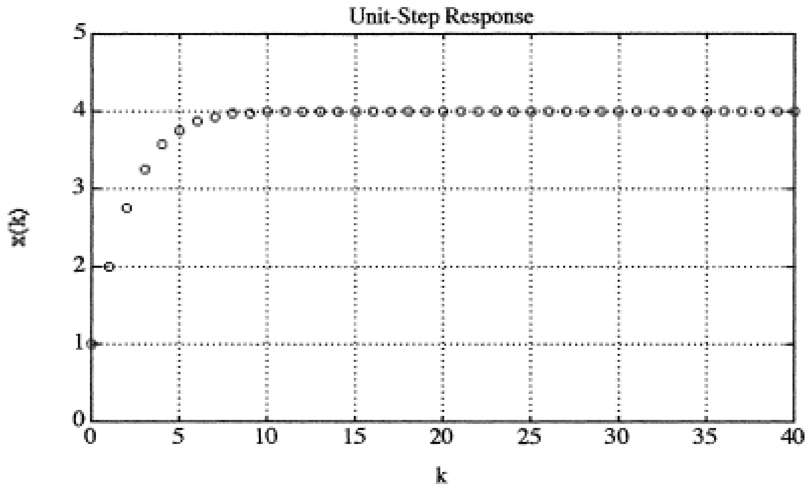
$$X(z) = \frac{z^3}{(z-1)(z^2 - z + 0.25)} = \frac{4z}{z-1} - \frac{3z}{z-0.5} - \frac{0.5z}{(z-0.5)^2}$$

Hence

$$x(k) = 4 - (3+k)(0.5)^k \qquad \text{for } k = 0, 1, 2, \ldots$$

Computational solution with MATLAB:

```
»% MATLAB Program for Problem B-2-17
»
»% ----- Unit-step repone -----
»
»num = [1  0  0];
»den = [1  -1  0.25];
»u = ones(1,41);
»v = [0  40  0  5];
»axis(v);
»k = 0:40;
»x = filter(num,den,u);
»plot(k,x,'o')
»grid
»title('Unit-Step Response')
»xlabel('k')
»ylabel('x(k)')
```



Unit-Step Response
5
4
3
2
1
0
x(k)
0
5
10
15
20
25
30
35
40
k