ECED 4601 Digital Control Systems

Midterm Reference Solution

Date: October 27 2016 **Room: Sexton Gym**

Time: 2:30PM-5:30PM

Student name	
ID	

1) Obtain the inverse z transform of $X(z)=\dfrac{z^{-3}}{(1-z^{-1})(1-0.2z^{-1})}$

a) Using inversion integral method

$X(z)z^{k-1}=\dfrac{z^k}{z^2(z-1)(z-0.2)}$ has double pole at z=0, single pole z=1 and z=0.2

$$K0 = residue \ \ at \ \ pole \ 0$$
$$= \lim_{z\to 0}\frac{d}{dz}\left(z^2\frac{z^k}{z^2(z-1)(z-0.2)}\right)$$
$$= \lim_{z\to 0}\frac{d}{dz}\left(\frac{z^k}{(z-1)(z-0.2)}\right)$$
$$=0$$

$$K1 = residue \ \ at \ \ pole \ 1$$
$$= \lim_{z\to 1}(z-1)\frac{z^k}{z^2(z-1)(z-0.2)} = 1.25$$
$$K2 = residue \ \ at \ \ pole \ 0.2$$
$$= \lim_{z\to 0.2}(z-0.2)\frac{z^k}{z^2(z-1)(z-0.2)} = -1.25(0.2)^{k-2}$$

$$x(k)=1.25-1.25(0.2)^{k-2}$$

b) Using partial-fraction expansion method

$$X(z) = \frac{z^{-3}}{(1-z^{-1})(1-0.2z^{-1})} = \frac{1}{z(z-1)(z-0.2)}$$
$$= \frac{5}{z} + \frac{1.25}{z-1} - \frac{6.25}{z-0.2}$$

Hence

$$x(k) = 5\,\delta_0(k-1) + 1.25 - 6.25(0.2)^{k-1} \qquad \text{for } k = 1, 2, 3, \ldots$$

That is,

$$x(k) = 0 \qquad \text{for } k = 0, 1, 2$$
$$= 1.25(1 - 0.2^{k-2}) \qquad \text{for } k = 3, 4, 5, \ldots$$

2) Find the solution of the following difference equation in a closed form.

$$x(k+2) = x(k+1) + x(k)$$

Where $x(0) = 0,\ x(1) = 1$

Solution:

By taking the z transform of this difference equation, we obtain

$$z^2 X(z) - z^2 x(0) - zx(1) = zX(z) - zx(0) + X(z)$$

Solving for $X(z)$ gives

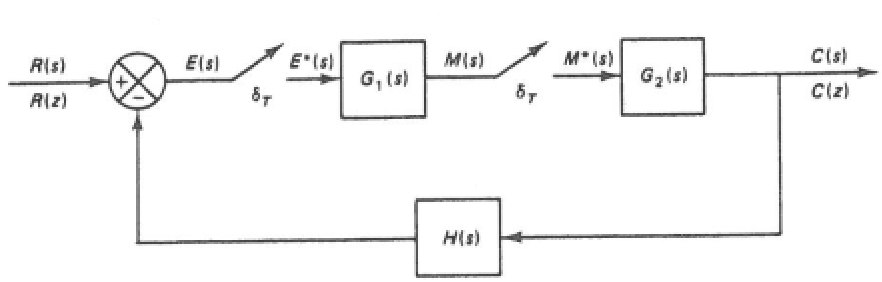
$$X(z) = \frac{z^2 x(0) + zx(1) - zx(0)}{z^2 - z - 1} = \frac{z}{z^2 - z - 1} = \frac{1}{\sqrt{5}}\left(\frac{z}{z - \frac{1+\sqrt{5}}{2}} - \frac{z}{z - \frac{1-\sqrt{5}}{2}}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1}{1 - \frac{1+\sqrt{5}}{2} z^{-1}} - \frac{1}{1 - \frac{1-\sqrt{5}}{2} z^{-1}}\right)$$

The inverse z transform of $X(z)$ is

$$x(k) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\left(\frac{1-\sqrt{5}}{2}\right)^k\right)\right),\ k=0,1,2,\ldots$$

3) Consider the system shown in figure. Obtain the close-loop pulse transfer function $\frac{C(z)}{R(z)}$. Also obtain the expression for $C(s)$



Solution:

$$C(s) = G_2(s)M^*(s)$$
$$M(s) = G_1(s)E^*(s)$$
$$E(s) = R(s) - H(s)C(s) = R(s) - H(s)G_2(s)M^*(s)$$

Taking the starred Laplace transforms of both sides of the last three equations gives

$$C^*(s) = G_2^*(s)M^*(s)$$
$$M^*(s) = G_1^*(s)E^*(s)$$
$$E^*(s) = R^*(s) - HG_2^*(s)M^*(s)$$

Solving for $C^*(s)$ gives

$$C^*(s) = G_2^*(s)G_1^*(s)[R^*(s) - HG_2^*(s)M^*(s)]$$

or

$$C^*(s) = G_1^*(s)G_2^*(s)R^*(s) - G_1^*(s)G_2^*(s)HG_2^*(s)M^*(s)$$
$$= G_1^*(s)G_2^*(s)R^*(s) - G_1^*(s)HG_2^*(s)C^*(s)$$

Thus,

$$C^*(s)[1 + G_1^*(s)HG_2^*(s)] = G_1^*(s)G_2^*(s)R^*(s)$$

or

$$\frac{C^*(s)}{R^*(s)} = \frac{G_1^*(s)G_2^*(s)}{1 + G_1^*(s)HG_2^*(s)}$$

In terms of the z transform notation, we have

$$\frac{C(z)}{R(z)} = \frac{G_1(z)G_2(z)}{1 + G_1(z)HG_2(z)}$$

The continuous-time output $C(s)$ can be obtained from the following equation:

$$C(s) = G_2(s)M^*(s) = G_2(s)\frac{G_1^*(s)R^*(s)}{1 + G_1^*(s)HG_2^*(s)}$$

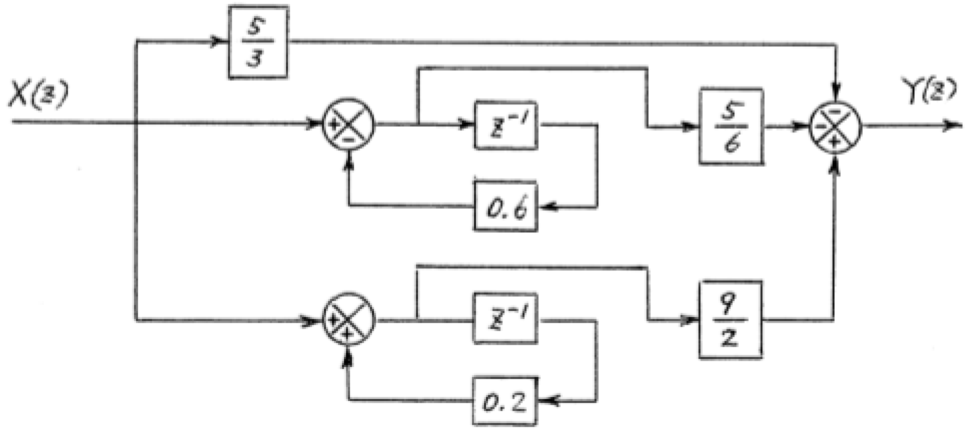
4) Consider the digital filter defined by

$$G(z) = \frac{2 + 2.2z^{-1} + 0.2z^{-2}}{1 + 0.4z^{-1} - 0.12z^{-2}}$$

Realize the digital filter in

a) The parallel scheme,

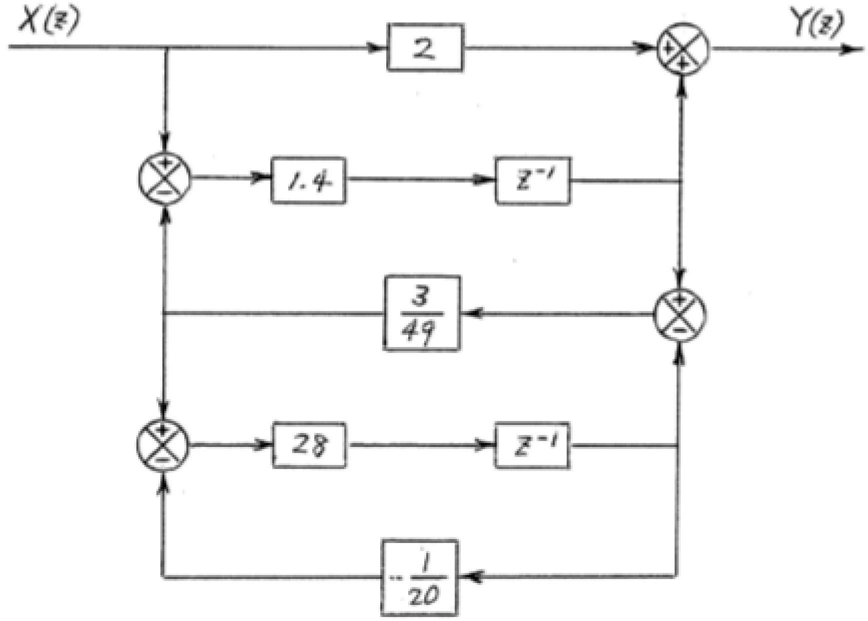
$$G(z) = -\frac{5}{3} - \frac{\frac{5}{6}}{1 + 0.6z^{-1}} + \frac{\frac{9}{2}}{1 - 0.2z^{-1}}$$



b) The ladder scheme.

$$G(z) = \frac{2z^2 + 2.2z + 0.2}{z^2 + 0.4z - 0.12}$$

$$= 2 + \cfrac{1}{\cfrac{1}{1.4} z + \cfrac{1}{\cfrac{49}{3} + \cfrac{1}{\cfrac{1}{28} z - \cfrac{1}{20}}}}$$



5) Consider the system described by

$$y(k) - 0.6y(k-1) - 0.81y(k-2) + 0.67y(k-3) - 0.12y(k-4) = x(k)$$

Determine the stability of the system:

Critically Stable.

Laplace Transform Pairs for Bilateral

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\text{Re}(s) > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\text{Re}(s) < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\text{Re}(s) > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\text{Re}(s) < 0$
6	$e^{-at}u(t)$	$\frac{1}{s+a}$	$\text{Re}(s) > -a$
7	$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\text{Re}(s) < -a$
8	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	$\text{Re}(s) > -a$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$\frac{1}{(s+a)^n}$	$\text{Re}(s) < -a$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos w_0 t]u(t)$	$\frac{s}{s^2+w_0^2}$	$\text{Re}(s) > 0$
12	$[\sin w_0 t]u(t)$	$\frac{w_0}{s^2+w_0^2}$	$\text{Re}(s) > 0$
13	$[e^{-at}\cos w_0 t]u(t)$	$\frac{s+a}{(s+a)^2+w_0^2}$	$\text{Re}(s) > -a$
14	$[e^{-at}\sin w_0 t]u(t)$	$\frac{w_0}{(s+a)^2+w_0^2}$	$\text{Re}(s) > -a$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = u(t) * \cdots * u(t)$	$\frac{1}{s^n}$	$\text{Re}(s) > 0$

Laplace Transform Property for Bilateral

Property	Signal	Laplace Transform	ROC
	$x(t)$, $x_1(t)$, $x_2(t)$	$X(s)$, $X_1(s)$, $X_2(s)$	R, R_1, R_2
Linearity	$ax_1(t)+bx_2(t)$	$aX_1(S)+bX_2(S)$	$R_1 \cap R_2$
Time Shifting	$x(t-t_0)$	$e^{-st_0}X(S)$	R
Shift in the s-Domain	$e^{s_0 t}x(t)$	$X(s-s_0)$	R
Time Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{s}{a})$	Scaled ROC
Conjugation	$x^*(t)$	$X^*(S^*)$	R
Convolution	$x_1(t)*x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the time-Domain	$\frac{dx(t)}{dt}$	$SX(S)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{dX(S)}{dS}$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{S}X(S)$	At least $R \cap \{\mathrm{Re}(s)>0\}$
Initial value theorem	$x(0^+)=\lim_{s\to\infty} SX(S)$		
Final value theorem	$\lim_{t\to\infty} x(t)=\lim_{s\to 0} SX(S)$		

Laplace Transform Property for Unilateral

Property	Signal	Laplace Transform
	$x(t)$, $x_1(t)$, $x_2(t)$	$\chi(s)$, $\chi_1(s)$, $\chi_2(s)$
Linearity	$ax_1(t)+bx_2(t)$	$a\chi_1(S)+b\chi_2(S)$
Shift in the s-Domain	$e^{s_0 t}x(t)$	$\chi(s-s_0)$
Time Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}\chi(\frac{s}{a})$
Conjugation	$x^*(t)$	$\chi^*(S)$
Convolution	$x_1(t)*x_2(t)$	$\chi_1(s)\chi_2(s)$
Differentiation in the time-Domain	$\frac{dx(t)}{dt}$	$S\chi(S)-x(0^-)$
	$\frac{d^2x(t)}{dt^2}$	$S^2\chi(S)-sx(0^-)-x'(0^-)$
Differentiation in the s-Domain	$-tx(t)$	$\frac{d\chi(S)}{dS}$
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{S}\chi(S)$
Initial value theorem	$x(0^+)=\lim_{s\to\infty} S\chi(S)$	
Final value theorem	$\lim_{t\to\infty} x(t)=\lim_{s\to 0} S\chi(S)$	

Table of z transforms 1

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	—	—	Kronecker delta $\delta_0(k)$ $1, \quad k = 0$ $0, \quad k \neq 0$	1
2.	—	—	$\delta_0(n-k)$ $1, \quad n = k$ $0, \quad n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1-z^{-1}}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$
8.	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$1-e^{-akT}$	$\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
11.	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)e^{-akT}$	$\frac{1-(1+aT)e^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$

Table of z transforms 2

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
12.	$\frac{2}{(s+a)^3}$	t^2e^{-at}	$(kT)^2e^{-akT}$	$\frac{T^2e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$	$akT-1+e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
18.			a^k	$\frac{1}{1-az^{-1}}$
19.			a^{k-1} $k=1,2,3,\ldots$	$\frac{z^{-1}}{1-az^{-1}}$
20.			ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21.			k^2a^{k-1}	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.			k^3a^{k-1}	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23.			k^4a^{k-1}	$\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24.			$a^k\cos k\pi$	$\frac{1}{1+az^{-1}}$
25.			$\frac{k(k-1)}{2!}$	$\frac{z^{-2}}{(1-z^{-1})^3}$
26.		$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}$		$\frac{z^{-m+1}}{(1-z^{-1})^m}$
27.			$\frac{k(k-1)}{2!}a^{k-2}$	$\frac{z^{-2}}{(1-az^{-1})^3}$
28.	$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}a^{k-m+1}$			$\frac{z^{-m+1}}{(1-az^{-1})^m}$

Property table:

	$x(t)$ or $x(k)$	$\mathcal{Z}[x(t)]$ or $\mathcal{Z}[x(k)]$
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t) + bx_2(t)$	$aX_1(z) + bX_2(z)$
3.	$x(t+T)$ or $x(k+1)$	$zX(z) - zx(0)$
4.	$x(t+2T)$	$z^2X(z) - z^2x(0) - zx(T)$
5.	$x(k+2)$	$z^2X(z) - z^2x(0) - zx(1)$
6.	$x(t+kT)$	$z^kX(z) - z^kx(0) - z^{k-1}x(T) - \cdots - zx(kT-T)$
7.	$x(t-kT)$	$z^{-k}X(z)$
8.	$x(n+k)$	$z^kX(z) - z^kx(0) - z^{k-1}x(1) - \cdots - zx(k-1)$
9.	$x(n-k)$	$z^{-k}X(z)$
10.	$tx(t)$	$-Tz\frac{d}{dz}X(z)$
11.	$kx(k)$	$-z\frac{d}{dz}X(z)$
12.	$e^{-at}x(t)$	$X(ze^{aT})$
13.	$e^{-ak}x(k)$	$X(ze^{a})$
14.	$a^kx(k)$	$X\left(\frac{z}{a}\right)$
15.	$ka^kx(k)$	$-z\frac{d}{dz}X\left(\frac{z}{a}\right)$
16.	$x(0)$	$\lim_{z\to\infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim_{z\to 1} [(1-z^{-1})X(z)]$ if $(1-z^{-1})X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k) = x(k) - x(k-1)$	$(1-z^{-1})X(z)$
19.	$\Delta x(k) = x(k+1) - x(k)$	$(z-1)X(z) - zx(0)$
20.	$\sum_{k=0}^{n} x(k)$	$\frac{1}{1-z^{-1}}X(z)$
21.	$\frac{\partial}{\partial a}x(t,a)$	$\frac{\partial}{\partial a}X(z,a)$
22.	$k^mx(k)$	$\left(-z\frac{d}{dz}\right)^m X(z)$
23.	$\sum_{k=0}^{n} x(kT)y(nT-kT)$	$X(z)Y(z)$
24.	$\sum_{k=0}^{\infty} x(k)$	$X(1)$

Table: z transform of $x(k+m)$ and $x(k-m)$

Discrete function	z Transform
$x(k+4)$	$z^4 X(z) - z^4 x(0) - z^3 x(1) - z^2 x(2) - zx(3)$
$x(k+3)$	$z^3 X(z) - z^3 x(0) - z^2 x(1) - zx(2)$
$x(k+2)$	$z^2 X(z) - z^2 x(0) - zx(1)$
$x(k+1)$	$zX(z) - zx(0)$
$x(k)$	$X(z)$
$x(k-1)$	$z^{-1} X(z)$
$x(k-2)$	$z^{-2} X(z)$
$x(k-3)$	$z^{-3} X(z)$
$x(k-4)$	$z^{-4} X(z)$