

# **ECED 4601 Digital Control Systems**

## **Final Exam**

**Date: December 10 2016**  
**Time: 8:30am-11:30am**

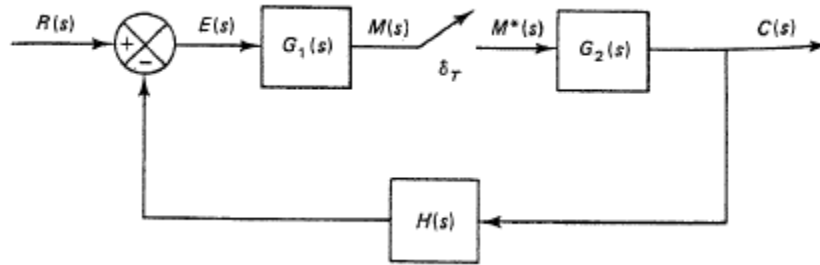
**Room: Sexton GYM**

**Each problem carries equal value**  
**Please do eight out of nine problems**  
**Please circle the problem you want to skip.**  
**By default, problem 9 will be skipped.**

<b>Student name</b>	
<b>ID</b>	
<b>Circle the problem you want to skip</b>	
<b>1 2 3 4 5 6 7 8 9</b>	

1) Obtain the inverse z transform of  $X(Z) = \frac{z^{-2}}{(1-z^{-1})^3}$  in a closed form. Detailed procedure is required.

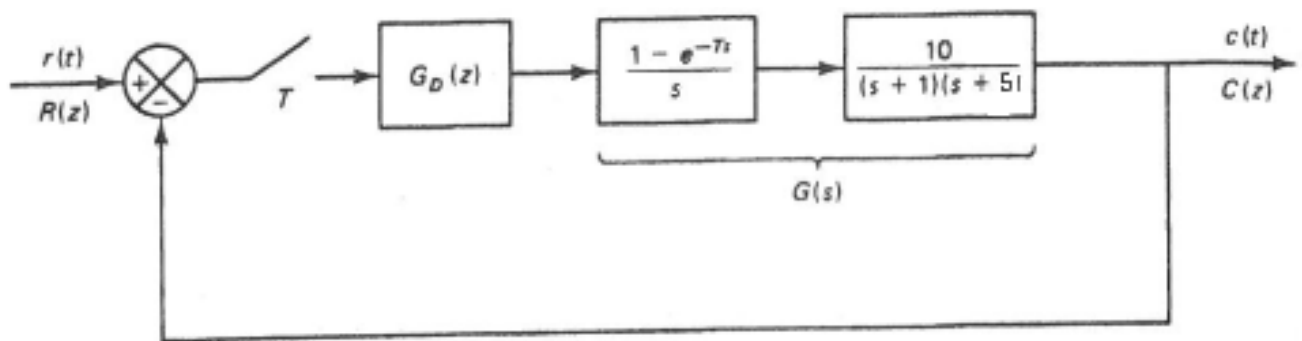
2) Obtain the discrete-time output  $c(z)$  of the closed-loop control system shown in figure. Also obtain the continuous time output  $c(s)$



3) Consider the system in following figure. A digital controller is designed and given as below.  $T=0.2$

$$G_D(z) = 1.4337 \frac{(z - 0.8187)}{(z - 1)}$$

- Find out the closed loop pulse transfer function.
- Determine the static velocity error constant  $K_V$



- 4) Find out whether the following quadratic form is PD or not PD

$$V(x) = 4x_1^2 + 4x_2^2 + x_3^2 + 2x_1x_2 - 2x_2x_3 - 4x_1x_3$$

- 5) Obtain the state transition matrix  $\psi(k)$  for following discrete time state equation.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

- 6) Consider the pulse transfer function system

$$G(z) = \frac{z+1}{z^2+z+0.16}$$

Obtain the state-space representation of the system in the following forms:

1. Controllable canonical form
2. Observable canonical form
3. Diagonal canonical form

- 7) Consider the discrete-time double integrator system defined by the equations:

$$x(k+1) = Gx(k) + Hu(k)$$

$$y(k) = Cx(k)$$

$$\text{Where } G = \begin{bmatrix} 0 & 0.2 \\ 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix}, \quad C = [1 \quad 0],$$

Design a state feedback gain matrix  $k$  such that the closed loop poles are located at  $z_1 = 0.6 + j0.4$  and  $z_2 = 0.6 - j0.4$

8) Consider the discrete-time system defined by the equations:

$$x(k+1) = Gx(k) + Hu(k)$$

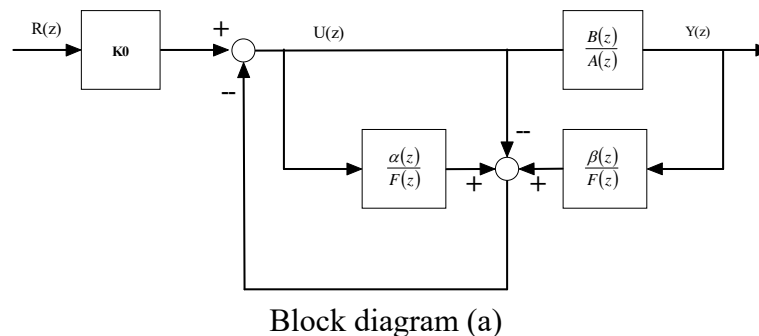
$$y(k) = Cx(k)$$

$$\text{Where } G = \begin{bmatrix} 0 & 0 & -0.25 \\ 1 & 0 & 0 \\ 0 & 1 & 0.5 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = [1 \ 0 \ 0],$$

Design a control system using following block diagram. For the pole placement part, we want to have three closed-loop poles at the origin, or

$H(z) = z^3$  and for the characteristic equation for the minimum-order observer, we want to have  $F(z) = z^2$

Use the polynomial equations approach to the design.



9) Consider the system:

$$\frac{Y(z)}{U(z)} = \frac{B(z)}{A(z)} = \frac{0.01873(z + 0.9356)}{(z-1)(z-0.8187)}$$

Using the polynomial equations approach, design a control system such that the system will behave like the following model,

$$\text{Where } G_{\text{Model}} = \frac{0.32}{z^3 - 1.2z + 0.52},$$

# Laplace Transform Pairs for Bilateral

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\text{Re}(s) > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\text{Re}(s) < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\text{Re}(s) > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\text{Re}(s) < 0$
6	$e^{-at}u(t)$	$\frac{1}{s+a}$	$\text{Re}(s) > -a$
7	$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\text{Re}(s) < -a$
8	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	$\text{Re}(s) > -a$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$\frac{1}{(s+a)^n}$	$\text{Re}(s) < -a$
10	$\delta(t-T)$	$e^{-sT}$	All s
11	$[\cos w_0 t]u(t)$	$\frac{s}{s^2 + w_0^2}$	$\text{Re}(s) > 0$
12	$[\sin w_0 t]u(t)$	$\frac{w_0}{s^2 + w_0^2}$	$\text{Re}(s) > 0$
13	$[e^{-at} \cos w_0 t]u(t)$	$\frac{s+a}{(s+a)^2 + w_0^2}$	$\text{Re}(s) > -a$
14	$[e^{-at} \sin w_0 t]u(t)$	$\frac{w_0}{(s+a)^2 + w_0^2}$	$\text{Re}(s) > -a$
15	$u_n(t) = \frac{d^n \delta(t)}{dt^n}$	$s^n$	All s
16	$u_{-n}(t) = u(t) * \dots * u(t)$	$\frac{1}{s^n}$	$\text{Re}(s) > 0$

## Laplace Transform Property for Bilateral

Property	Signal	Laplace Transform	ROC
	$x(t), x_1(t), x_2(t)$	$X(s), X_1(s), X_2(s)$	$R, R_1, R_2$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(S) + bX_2(S)$	$R_1 \cap R_2$
Time Shifting	$x(t - t_0)$	$e^{-st_0} X(S)$	$R$
Shift in the s-Domain	$e^{s_0 t} x(t)$	$X(s - s_0)$	$R$
Time Scaling	$x(at)$	$\frac{1}{ a } X\left(\frac{s}{a}\right)$	Scaled ROC
Conjugation	$x^*(t)$	$X^*(S^*)$	$R$
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the time-Domain	$\frac{dx(t)}{dt}$	$SX(S)$	At least $R$
Differentiation in the s-Domain	$-tx(t)$	$\frac{dX(S)}{dS}$	$R$
Integration in the Time Domain	$\int_{-\infty}^t x(\tau) d\tau$	$\frac{1}{S} X(S)$	At least $R \cap \{\text{Re}(s) > 0\}$
Initial value theorem	$x(0^+) = \lim_{s \rightarrow \infty} SX(S)$		
Final value theorem	$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} SX(S)$		

## Laplace Transform Property for Unilateral

Property	Signal	Laplace Transform
	$x(t), x_1(t), x_2(t)$	$\chi(s), \chi_1(s), \chi_2(s)$
Linearity	$ax_1(t) + bx_2(t)$	$a\chi_1(S) + b\chi_2(S)$
Shift in the s-Domain	$e^{s_0 t} x(t)$	$\chi(s - s_0)$
Time Scaling	$x(at)$	$\frac{1}{ a } \chi\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$\chi^*(S)$
Convolution	$x_1(t) * x_2(t)$	$\chi_1(s)\chi_2(s)$
Differentiation in the time-Domain	$\frac{dx(t)}{dt}$	$S\chi(S) - x(0^-)$
	$\frac{d^2 x(t)}{dt^2}$	$S^2 \chi(S) - sx(0^-) - x'(0^-)$
Differentiation in the s-Domain	$-tx(t)$	$\frac{d\chi(S)}{dS}$

Integration in the Time Domain	$\int_{-\infty}^t x(\tau) d\tau$	$\frac{1}{S} \chi(S)$
Initial value theorem	$x(0^+) = \lim_{s \rightarrow \infty} S\chi(S)$	
Final value theorem	$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} S\chi(S)$	

## Table of z transforms 1

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	—	—	Kronecker delta $\delta_b(k)$ 1, $k = 0$ 0, $k \neq 0$	1
2.	—	—	$\delta_b(n - k)$ 1, $n = k$ 0, $n \neq k$	$z^{-k}$
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1 - z^{-1}}$
4.	$\frac{1}{s + a}$	$e^{-at}$	$e^{-akT}$	$\frac{1}{1 - e^{-aT} z^{-1}}$
5.	$\frac{1}{s^2}$	$t$	$kT$	$\frac{Tz^{-1}}{(1 - z^{-1})^2}$
6.	$\frac{2}{s^3}$	$t^2$	$(kT)^2$	$\frac{T^2 z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$
7.	$\frac{6}{s^4}$	$t^3$	$(kT)^3$	$\frac{T^3 z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}$
8.	$\frac{a}{s(s + a)}$	$1 - e^{-at}$	$1 - e^{-akT}$	$\frac{(1 - e^{-aT})z^{-1}}{(1 - z^{-1})(1 - e^{-aT} z^{-1})}$
9.	$\frac{b - a}{(s + a)(s + b)}$	$e^{-at} - e^{-bt}$	$e^{-akT} - e^{-bkT}$	$\frac{(e^{-aT} - e^{-bT})z^{-1}}{(1 - e^{-aT} z^{-1})(1 - e^{-bT} z^{-1})}$
10.	$\frac{1}{(s + a)^2}$	$te^{-at}$	$kTe^{-akT}$	$\frac{Tze^{-aT} z^{-1}}{(1 - e^{-aT} z^{-1})^2}$
11.	$\frac{s}{(s + a)^2}$	$(1 - at)e^{-at}$	$(1 - akT)e^{-akT}$	$\frac{1 - (1 + aT)e^{-aT} z^{-1}}{(1 - e^{-aT} z^{-1})^2}$

## Table of z transforms 2

	X(s)	x(t)	x(kT) or x(k)	X(z)
12.	$\frac{2}{(s+a)^3}$	$t^2 e^{-at}$	$(kT)^2 e^{-akT}$	$\frac{T^2 e^{-aT}(1 + e^{-aT}z^{-1})z^{-1}}{(1 - e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at - 1 + e^{-at}$	$akT - 1 + e^{-akT}$	$\frac{[(aT - 1 + e^{-aT}) + (1 - e^{-aT} - aTe^{-aT})z^{-1}]z^{-1}}{(1 - z^{-1})^2(1 - e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2 + \omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1} \sin \omega T}{1 - 2z^{-1} \cos \omega T + z^{-2}}$
15.	$\frac{s}{s^2 + \omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1 - z^{-1} \cos \omega T}{1 - 2z^{-1} \cos \omega T + z^{-2}}$
16.	$\frac{\omega}{(s+a)^2 + \omega^2}$	$e^{-at} \sin \omega t$	$e^{-akT} \sin \omega kT$	$\frac{e^{-aT} z^{-1} \sin \omega T}{1 - 2e^{-aT} z^{-1} \cos \omega T + e^{-2aT} z^{-2}}$
17.	$\frac{s+a}{(s+a)^2 + \omega^2}$	$e^{-at} \cos \omega t$	$e^{-akT} \cos \omega kT$	$\frac{1 - e^{-aT} z^{-1} \cos \omega T}{1 - 2e^{-aT} z^{-1} \cos \omega T + e^{-2aT} z^{-2}}$
18.			$a^k$	$\frac{1}{1 - az^{-1}}$
19.			$a^{k-1}$ $k = 1, 2, 3, \dots$	$\frac{z^{-1}}{1 - az^{-1}}$
20.			$ka^{k-1}$	$\frac{z^{-1}}{(1 - az^{-1})^2}$
21.			$k^2 a^{k-1}$	$\frac{z^{-1}(1 + az^{-1})}{(1 - az^{-1})^3}$
22.			$k^3 a^{k-1}$	$\frac{z^{-1}(1 + 4az^{-1} + a^2 z^{-2})}{(1 - az^{-1})^4}$
23.			$k^4 a^{k-1}$	$\frac{z^{-1}(1 + 11az^{-1} + 11a^2 z^{-2} + a^3 z^{-3})}{(1 - az^{-1})^5}$
24.			$a^k \cos k\pi$	$\frac{1}{1 + az^{-1}}$
25.			$\frac{k(k-1)}{2!}$	$\frac{z^{-2}}{(1 - z^{-1})^3}$
26.			$\frac{k(k-1)\dots(k-m+2)}{(m-1)!}$	$\frac{z^{-m+1}}{(1 - z^{-1})^m}$
27.			$\frac{k(k-1)}{2!} a^{k-2}$	$\frac{z^{-2}}{(1 - az^{-1})^3}$
28.			$\frac{k(k-1)\dots(k-m+2)}{(m-1)!} a^{k-m+1}$	$\frac{z^{-m+1}}{(1 - az^{-1})^m}$



## Property table:

	$x(t)$ or $x(k)$	$\mathcal{Z}[x(t)]$ or $\mathcal{Z}[x(k)]$
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t) + bx_2(t)$	$aX_1(z) + bX_2(z)$
3.	$x(t + T)$ or $x(k + 1)$	$zX(z) - zx(0)$
4.	$x(t + 2T)$	$z^2X(z) - z^2x(0) - zx(T)$
5.	$x(k + 2)$	$z^2X(z) - z^2x(0) - zx(1)$
6.	$x(t + kT)$	$z^kX(z) - z^kx(0) - z^{k-1}x(T) - \dots - zx(kT - T)$
7.	$x(t - kT)$	$z^{-k}X(z)$
8.	$x(n + k)$	$z^kX(z) - z^kx(0) - z^{k-1}x(1) - \dots - zx(k - 1)$
9.	$x(n - k)$	$z^{-k}X(z)$
10.	$tx(t)$	$-Tz \frac{d}{dz}X(z)$
11.	$kx(k)$	$-z \frac{d}{dz}X(z)$
12.	$e^{-at}x(t)$	$X(ze^{aT})$
13.	$e^{-ak}x(k)$	$X(ze^a)$
14.	$a^kx(k)$	$X\left(\frac{z}{a}\right)$
15.	$ka^kx(k)$	$-z \frac{d}{dz}X\left(\frac{z}{a}\right)$
16.	$x(0)$	$\lim_{z \rightarrow \infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim_{z \rightarrow 1} [(1 - z^{-1})X(z)]$ if $(1 - z^{-1})X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k) = x(k) - x(k - 1)$	$(1 - z^{-1})X(z)$
19.	$\Delta x(k) = x(k + 1) - x(k)$	$(z - 1)X(z) - zx(0)$
20.	$\sum_{k=0}^n x(k)$	$\frac{1}{1 - z^{-1}}X(z)$
21.	$\frac{\partial}{\partial a}x(t, a)$	$\frac{\partial}{\partial a}X(z, a)$
22.	$k^m x(k)$	$\left(-z \frac{d}{dz}\right)^m X(z)$
23.	$\sum_{k=0}^n x(kT)y(nT - kT)$	$X(z)Y(z)$
24.	$\sum_{k=0}^{\infty} x(k)$	$X(1)$

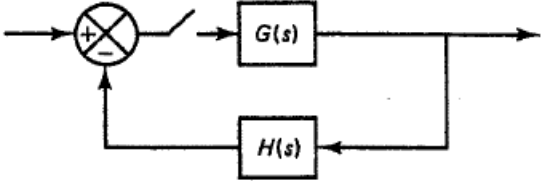
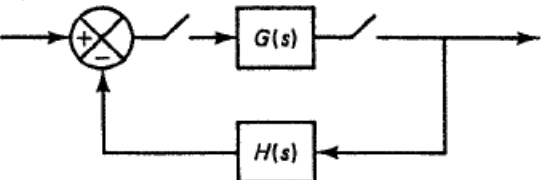
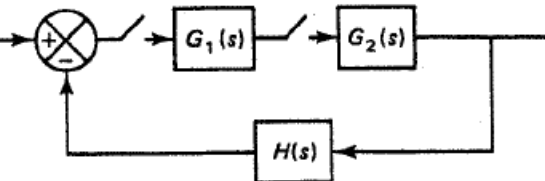
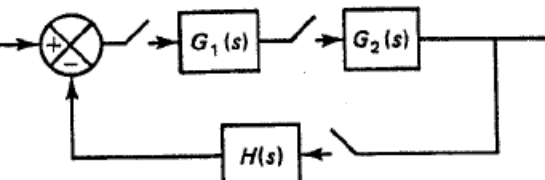
Table: z transform of  $x(k+m)$  and  $x(k-m)$

Discrete function	z Transform
$x(k+4)$	$z^4 X(z) - z^4 x(0) - z^3 x(1) - z^2 x(2) - zx(3)$
$x(k+3)$	$z^3 X(z) - z^3 x(0) - z^2 x(1) - zx(2)$
$x(k+2)$	$z^2 X(z) - z^2 x(0) - zx(1)$
$x(k+1)$	$zX(z) - zx(0)$
$x(k)$	$X(z)$
$x(k-1)$	$z^{-1} X(z)$
$x(k-2)$	$z^{-2} X(z)$
$x(k-3)$	$z^{-3} X(z)$
$x(k-4)$	$z^{-4} X(z)$

Jury table is given as follows

Row	$z^0$	$z^1$	$z^2$	$z^3$	...	$z^{n-2}$	$z^{n-1}$	$z^n$
1	$a_n$	$a_{n-1}$	$a_{n-2}$	$a_{n-3}$	...	$a_2$	$a_1$	$a_0$
2	$a_0$	$a_1$	$a_2$	$a_3$	...	$a_{n-2}$	$a_{n-1}$	$a_n$
3	$b_{n-1}$	$b_{n-2}$	$b_{n-3}$	$b_{n-4}$	...	$b_1$	$b_0$	
4	$b_0$	$b_1$	$b_2$	$b_3$	...	$b_{n-2}$	$b_{n-1}$	
5	$c_{n-2}$	$c_{n-3}$	$c_{n-4}$	$c_{n-5}$	...	$c_0$		
6	$c_0$	$c_1$	$c_2$	$c_3$	...	$c_{n-2}$		
.	.	.	.	.	.	.	.	.
$2n-5$	$p_3$	$p_2$	$p_1$	$p_0$				
$2n-4$	$p_0$	$p_1$	$p_2$	$p_3$				
$2n-3$	$q_2$	$q_1$	$q_0$					

# Static error constants for typical closed-loop configurations of discrete time control system

Closed-loop configuration	Values of $K_p$ , $K_v$ , and $K_a$
	$K_p = \lim_{z \rightarrow 1} GH(z)$ $K_v = \lim_{z \rightarrow 1} \frac{(1 - z^{-1})GH(z)}{T}$ $K_a = \lim_{z \rightarrow 1} \frac{(1 - z^{-1})^2 GH(z)}{T^2}$
	$K_p = \lim_{z \rightarrow 1} G(z)H(z)$ $K_v = \lim_{z \rightarrow 1} \frac{(1 - z^{-1})G(z)H(z)}{T}$ $K_a = \lim_{z \rightarrow 1} \frac{(1 - z^{-1})^2 G(z)H(z)}{T^2}$
	$K_p = \lim_{z \rightarrow 1} G_1(z)HG_2(z)$ $K_v = \lim_{z \rightarrow 1} \frac{(1 - z^{-1})G_1(z)HG_2(z)}{T}$ $K_a = \lim_{z \rightarrow 1} \frac{(1 - z^{-1})^2 G_1(z)HG_2(z)}{T^2}$
	$K_p = \lim_{z \rightarrow 1} G_1(z)G_2(z)H(z)$ $K_v = \lim_{z \rightarrow 1} \frac{(1 - z^{-1})G_1(z)G_2(z)H(z)}{T}$ $K_a = \lim_{z \rightarrow 1} \frac{(1 - z^{-1})^2 G_1(z)G_2(z)H(z)}{T^2}$

## 2 by 2 matrix inversion formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

## 3 by 3 matrix inverse formula

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det(\mathbf{A}) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{13}a_{21} - a_{11}a_{23} \\ a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}$$