ECED 4601 Digital Control Systems

Final Exam Reference Solution

Date: December 10 2016 **Room: Sexton GYM**
Time: 8:30am-11:30am

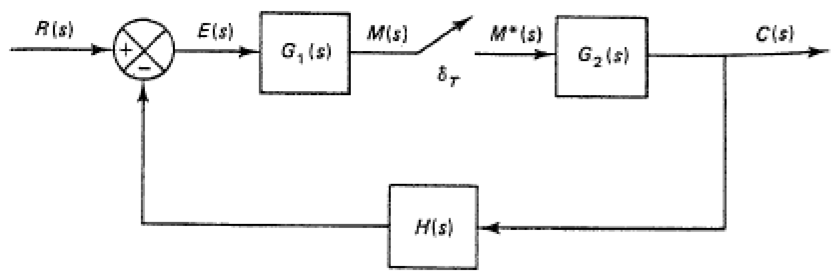
Each question carries equal value
Please do eight out of nine questions
Please circle the problem you will skip.

Student name	
ID	
Circle the problem you want to skip **1 2 3 4 5 6 7 8 9**	

1) Obtain the inverse z transform of $X(Z) = \frac{z^{-2}}{(1-z^{-1})^3}$ in a closed form. Detailed procedure is required.

$$
\begin{aligned}
X(k) &= residue\ \ of \frac{z^k}{(z-1)^3} at\ triple\ pole\ 1 \\
&= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3\frac{z^k}{(z-1)^3}\right) \\
&= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\ \left(z^k\right) \\
&= \frac{1}{2}k(k-1)
\end{aligned}
$$

2) Obtain the discrete-time output $c(z)$ of the closed-loop control system shown in figure. Also obtain the continuous time output $c(s)$



From the diagram we have

$$C(s) = G_2(s)M^*(s)$$
$$M(s) = G_1(s)E(s)$$
$$E(s) = R(s) - H(s)C(s)$$

Hence,

$$M(s) = G_1(s)[R(s) - H(s)C(s)]$$
$$= G_1(s)R(s) - G_1(s)H(s)G_2(s)M^*(s)$$

Taking the starred Laplace transform of this last equation, we obtain

$$M^*(s) = [G_1 R(s)]^* - [G_1 G_2 H(s)]^* M^*(s)$$

or

$$M^*(s) = \frac{[G_1 R(s)]^*}{1 + [G_1 G_2 H(s)]^*}$$

Since $C(s) = G_2(s)M^*(s)$, we have

$$C^*(s) = G_2^*(s)M^*(s) = \frac{G_2^*(s)[G_1 R(s)]^*}{1 + [G_1 G_2 H(s)]^*}$$

In terms of the z transform notation,

$$C(z) = \frac{G_2(z)G_1 R(z)}{1 + G_1 G_2 H(z)}$$

This last equation gives the discrete-time output $C(z)$.

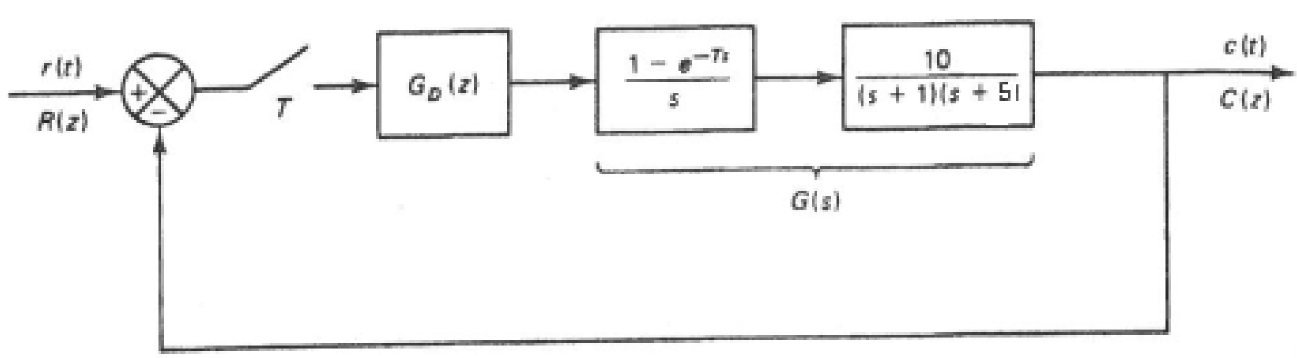
The continuous-time output $C(s)$ can be obtained from the following equation:

$$C(s) = G_2(s)M^*(s) = G_2(s)\frac{[G_1 R(s)]^*}{1 + [G_1 G_2 H(s)]^*}$$

3) Consider the system in figure. A digital control is designed and given as below. T=0..2

$$G_D(z) = 1.4337\frac{(z - 0.8187)}{(z - 1)}$$

a) Find out the closed loop pulse transfer function.
b) Determine the static velocity error constant Kv



$$G_D(z)G(z) = 1.4337\frac{(z-0.8187)}{(z-1)}0.1372\frac{(z+0.6706)}{(z-0.8187)(z-0.3679)}$$
$$= 0.1967\frac{z^{-1}(1+0.6706z^{-1})}{(1-z^{-1})(1-0.3679z^{-1})}$$

Close loop transfer function is

$$\frac{G_D(z)G(z)}{1+G_D(z)G(z)}$$

$$Kv = \lim_{z\to 1}\frac{1-z^{-1}}{0.2}0.1967\frac{z^{-1}(1+0.6706z^{-1})}{(1-z^{-1})(1-0.3679z^{-1})} = 2.599$$

4) Find out whether the following quadratic form is PD or not PD

$$V(x) = 4x_1^2 + 4x_2^2 + x_3^2 + 2x_1x_2 - 2x_2x_3 - 4x_1x_3$$

$$V(x) = x^T Px = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 4 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Check all the principal minor:

$4 > 0$, $\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} > 0$, $\begin{vmatrix} 4 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{vmatrix} < 0$, P is not PD thus $V(x)$ not PD

5) Obtain the state transition matrix $\psi(k)$ for following discrete time state equation.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(k)$$

$$\psi(k) = G^k = z^{-1}\left((zi - G)^{-1} z\right)$$

$$(zi-G)^{-1}=\begin{bmatrix} z & -1 \\ 0.16 & z+1 \end{bmatrix}^{-1}=\frac{1}{(z+0.2)(z+0.8)}\begin{bmatrix} z+1 & 1 \\ -0.16 & z \end{bmatrix}$$

$$(zi-G)^{-1}z=\begin{bmatrix} z & -1 \\ 0.16 & z+1 \end{bmatrix}^{-1}z=\frac{1}{(z+0.2)(z+0.8)}\begin{bmatrix} z+1 & 1 \\ -0.16 & z \end{bmatrix}$$

$$=\begin{bmatrix} \frac{4/3}{z+0.2}-\frac{1/3}{z+0.8} & \frac{5/3}{z+0.2}-\frac{5/3}{z+0.8} \\ \frac{-0.8/3}{z+0.2}+\frac{0.8/3}{z+0.8} & -\frac{1/3}{z+0.2}+\frac{4/3}{z+0.8} \end{bmatrix}z$$

$$=\begin{bmatrix} \frac{4/3}{1+0.2z^{-1}}-\frac{1/3}{1+0.8z^{-1}} & \frac{5/3}{1+0.2z^{-1}}-\frac{5/3}{1+0.8z^{-1}} \\ \frac{-0.8/3}{1+0.2z^{-1}}+\frac{0.8/3}{1+0.8z^{-1}} & -\frac{1/3}{1+0.2z^{-1}}+\frac{4/3}{1+0.8z^{-1}} \end{bmatrix}$$

$$\psi(k)=G^k=z^{-1}\left((zi-G)^{-1}z\right)=Z^{-1}\left(\begin{bmatrix} \frac{4/3}{1+0.2z^{-1}}-\frac{1/3}{1+0.8z^{-1}} & \frac{5/3}{1+0.2z^{-1}}-\frac{5/3}{1+0.8z^{-1}} \\ \frac{-0.8/3}{1+0.2z^{-1}}+\frac{0.8/3}{1+0.8z^{-1}} & -\frac{1/3}{1+0.2z^{-1}}+\frac{4/3}{1+0.8z^{-1}} \end{bmatrix}\right)$$

$$=\begin{pmatrix} \frac{4}{3}(-0.2)^k-\frac{1}{3}(-0.8)^k & \frac{5}{3}(-0.2)^k-\frac{5}{3}(-0.8)^k \\ \frac{-0.8}{3}(-0.2)^k+\frac{0.8}{3}(-0.8)^k & -\frac{1}{3}(-0.2)^k+\frac{4}{3}(-0.8)^k \end{pmatrix}$$

6) Consider the pulse transfer function system

$$G(z)=\frac{z+1}{z^2+z+0.16}$$

Obtain the state-space representation of the system in the following forms:

1. Controllable canonical form
2. Observable canonical form
3. Diagonal canonical form

1. *Controllable canonical form.*

$$a_1 = 1, \quad a_2 = 0.16, \quad b_0 = 0, \quad b_1 = 1, \quad b_2 = 1$$

Hence, referring to Equations (6–37) and (6–38), we obtain

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

2. *Observable canonical form.* Since $a_1 = 1$, $a_2 = 0.16$, $b_0 = 0$, $b_1 = 1$, and $b_2 = 1$, referring to Equations (6–41) and (6–42), we obtain

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -0.16 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [0 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

3. *Diagonal canonical form.* Notice that

$$G(z) = \frac{\frac{4}{3}}{z + 0.2} + \frac{-\frac{1}{3}}{z + 0.8}$$

By comparing this last equation with Equation (6–47), we obtain

$$\alpha_1 \beta_1 = \tfrac{4}{3}, \quad \alpha_2 \beta_2 = -\tfrac{1}{3}, \quad p_1 = -0.2, \quad p_2 = -0.8, \quad D = 0$$

Hence, by arbitrarily choosing $\alpha_1 = \alpha_2 = 1$
we obtain

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.8 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [\tfrac{4}{3} \quad -\tfrac{1}{3}] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

7) Consider the discrete-time double integrator system defined by the equations:

$$x(k+1) = Gx(k) + Hu(k)$$
$$y(k) = Cx(k)$$

Where $G = \begin{bmatrix} 0 & 0.2 \\ 0 & 1 \end{bmatrix}$, $\quad H = \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix}$, $\quad C = [1 \quad 0]$,

Design a state feedback gain matrix k such that the closed loop poles are located at $z_1 = 0.6 + j0.4$ and $z_2 = 0.6 + j0.4$

We shall first examine the controllability and observability of the system. Since the rank of the matrices

$$[\mathbf{H} \vdots \mathbf{GH}] = \begin{bmatrix} 0.02 & 0.06 \\ 0.2 & 0.2 \end{bmatrix}, \qquad [\mathbf{C}^* \vdots \mathbf{G}^*\mathbf{C}^*] = \begin{bmatrix} 1 & 1 \\ 0 & 0.2 \end{bmatrix}$$

is 2 in both cases, the system is completely state controllable and observable.

We shall now solve the pole placement portion of the problem. Since

$$|z\mathbf{I} - \mathbf{G}| = \begin{vmatrix} z-1 & -0.2 \\ 0 & z-1 \end{vmatrix} = z^2 - 2z + 1 = z^2 + a_1 z + a_2 = 0$$

we have

$$a_1 = -2, \qquad a_2 = 1$$

The desired characteristic equation is given by

$$|z\mathbf{I} - \mathbf{G} + \mathbf{HK}| = (z - 0.6 - j0.4)(z - 0.6 + j0.4) = z^2 - 1.2z + 0.52$$
$$= z^2 + \alpha_1 z + \alpha_2 = 0$$

Hence,

$$\alpha_1 = -1.2, \qquad \alpha_2 = 0.52$$

state feedback gain matrix **K** is obtained as follows:

$$\mathbf{K} = [\alpha_2 - a_2 \vdots \alpha_1 - a_1]\mathbf{T}^{-1} = [-0.48 \quad 0.8]\mathbf{T}^{-1}$$

where

$$\mathbf{T} = [\mathbf{H} \vdots \mathbf{GH}]\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.02 & 0.06 \\ 0.2 & 0.2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0.02 & 0.02 \\ -0.2 & 0.2 \end{bmatrix}$$

and

$$\mathbf{T}^{-1} = \begin{bmatrix} 25 & -2.5 \\ 25 & 2.5 \end{bmatrix}$$

Thus, the state feedback gain matrix **K** becomes

$$\mathbf{K} = [-0.48 \quad 0.8]\begin{bmatrix} 25 & -2.5 \\ 25 & 2.5 \end{bmatrix} = [8 \quad 3.2]$$

8) Consider the discrete-time system defined by the equations:

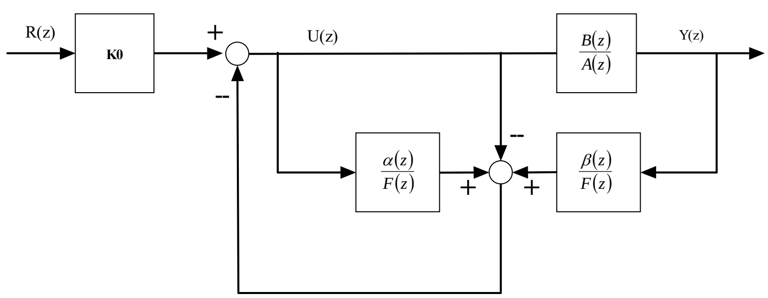
$$x(k+1) = Gx(k) + Hu(k)$$
$$y(k) = Cx(k)$$

Where $G = \begin{bmatrix} 0 & 0 & -0.25 \\ 1 & 0 & 0 \\ 0 & 1 & 0.5 \end{bmatrix}$, $H = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$,

Design a control system using following block diagram. For the pole placement part, we want to have three closed-loop poles at the origin, or

$H(z) = z^3$ and for the characteristic equation for the minimum-order observer, we want to have $F(z) = z^2$

Use the polynomial equations approach to the design.



Block diagram (a)

First, we determine the transfer function Y(z)/U(z).

$$\frac{Y(z)}{U(z)} = \underset{\sim}{C}(z\underset{\sim}{I} - \underset{\sim}{G})^{-1}\underset{\sim}{H}$$

$$= [1 \quad 0 \quad 0]\begin{bmatrix} z & 0 & 0.25 \\ -1 & z & 0 \\ 0 & -1 & z - 0.5 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{z^2 - 0.75z}{z^3 - 0.5z^2 + 0.25} = \frac{B(z)}{A(z)}$$

Hence

$$A(z) = z^3 - 0.5z^2 + 0.25$$

$$B(z) = z^2 - 0.75z$$

Thus,

$$a_1 = -0.5, \quad a_2 = 0, \quad a_3 = 0.25$$

$$b_0 = 0, \quad b_1 = 1, \quad b_2 = -0.75, \quad b_3 = 0$$

Next, we solve the following Diophantine equation:

$$\alpha(z)A(z) + \beta(z)B(z) = H(z)F(z)$$

or

$$\alpha(z)(z^3 - 0.5z^2 + 0.25) + \beta(z)(z^2 - 0.75z) = z^5$$

where

$$\alpha(z) = \alpha_0 z^2 + \alpha_1 z + \alpha_2$$

$$\beta(z) = \beta_0 z^2 + \beta_1 z + \beta_2$$

To determine $\alpha(z)$ and $\beta(z)$ we define Sylvester matrix $\underset{\sim}{E}$.

$$\underset{\sim}{E} = \begin{bmatrix} 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & -0.75 & 0 & 0 \\ -0.5 & 0 & 0.25 & 1 & -0.75 & 0 \\ 1 & -0.5 & 0 & 0 & 1 & -0.75 \\ 0 & 1 & -0.5 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

We also define matrices $\underset{\sim}{D}$ and $\underset{\sim}{M}$ as follows:

$$\underset{\sim}{D} = \begin{bmatrix} d_5 \\ d_4 \\ d_3 \\ d_2 \\ d_1 \\ d_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \underset{\sim}{M} = \begin{bmatrix} \alpha_2 \\ \alpha_1 \\ \alpha_0 \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix}$$

Then $\underset{\sim}{M}$ is determined from

$$\underset{\sim}{M} = \underset{\sim}{E}^{-1} \underset{\sim}{D}$$

A MATLAB solution for determining $\underset{\sim}{M}$ is shown below.

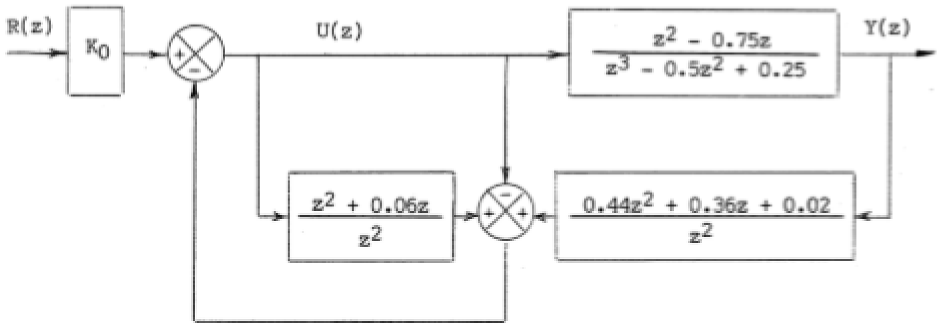
```
E =

    0.2500         0         0         0         0         0
         0    0.2500         0   -0.7500         0         0
   -0.5000         0    0.2500    1.0000   -0.7500         0
    1.0000   -0.5000         0         0    1.0000   -0.7500
         0    1.0000   -0.5000         0         0    1.0000
         0         0    1.0000         0         0         0

/ D = [0;0;0;0;0;1];
/ M = inv(E)*D

M =

         0
    0.0600
    1.0000
    0.0200
    0.3600
    0.4400
```

Thus, $\alpha(z)$ and $\beta(z)$ are determined as follows:

$$\alpha(z) = z^2 + 0.06z$$

$$\beta(z) = 0.44z^2 + 0.36z + 0.02$$

The block diagram for the designed system is shown on next page.



The gain constant K_0 is determined from the requirement that $y(\infty)$ is unity in the unit-step response. Since

$$\frac{Y(z)}{R(z)} = \frac{K_0 F(z)B(z)}{H(z)F(z)} = \frac{K_0 B(z)}{H(z)} = \frac{K_0(z - 0.75)}{z^2}$$

we have

$$\lim_{k \to \infty} y(k) = \lim_{z \to 1}\left(\frac{z-1}{z}\right)\left[\frac{K_0(z-0.75)}{z^2}\right]\left(\frac{z}{z-1}\right)$$

$$= 0.25K_0 = 1$$

Hence K_0 is determined as

$$K_0 = 4$$

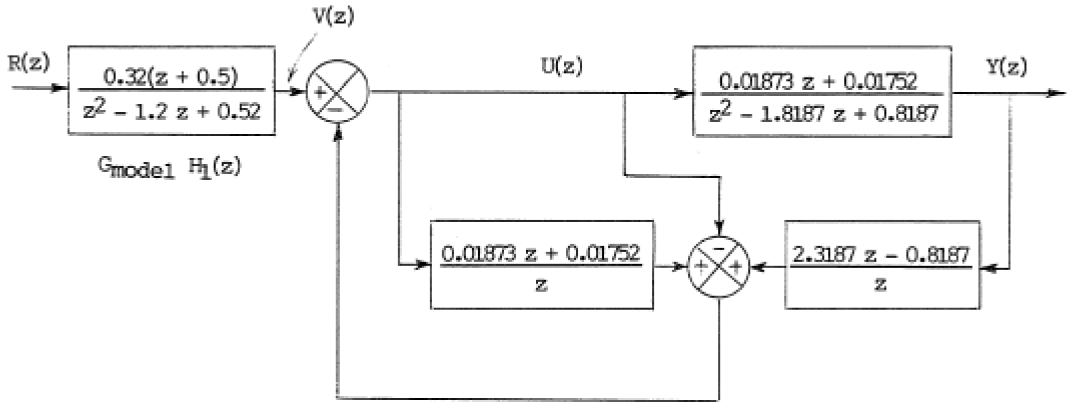
The designed system is of second order.

9) Consider the system:

$$\frac{Y(z)}{U(z)} = \frac{B(z)}{A(z)} = \frac{0.01873(z+0.9356)}{(z-1)(z-0.8187)}$$

Using the polynomial equations approach, design a control system such that the system will behave like the following model,

Where $G_{Model} = \dfrac{0.32}{z^3 - 1.2z + 0.52}$,



For the given plant,

$$\frac{Y(z)}{U(z)} = \frac{0.01873z + 0.01752}{z^2 - 1.8187z + 0.8187} = \frac{B(z)}{A(z)}$$

Thus,

$$A(z) = z^2 - 1.8187z + 0.8187 = z^2 + a_1 z + a_2$$

$$B(z) = 0.01873z + 0.01752 = b_0 z^2 + b_1 z + b_2$$

Hence,

$$a_1 = -1.8187, \quad a_2 = 0.8187$$

$$b_0 = 0, \quad b_1 = 0.01873, \quad b_2 = 0.01752$$

[Clearly, there are no common factors between $A(z)$ and $B(z)$ and the numerator $B(z)$ is a stable polynomial.]

In the design process we choose $H(z)$ as the desired characteristic polynomial of degree 2. Let us choose a stable polynomial of degree 1 as $H_1(z)$, or

$$H_1(z) = z + 0.5$$

[Choice of $H_1(z)$ is, in a sense, arbitrary as long as it is a stable polynomial.] Now define

$$H(z) = B(z)H_1(z) = (0.01873z + 0.01752)(z + 0.5)$$

A MATLAB solution for the determination of $\underset{\sim}{M}$ is given below.

```
E =

    0.8187         0    0.0175         0
   -1.8187    0.8187    0.0187    0.0175
    1.0000   -1.8187         0    0.0187
         0    1.0000         0         0

» D = [0;0.00876;0.026885;0.01873];
» format long
» M = inv(E)*D

M =

   0.01752000000000
   0.01873000000000
  -0.81870000000000
   2.31870000000000
```

Hence

$$\alpha(z) = \alpha_0 z + \alpha_1 = 0.01873z + 0.01752$$

$$\beta(z) = \beta_0 z + \beta_1 = 2.3187z - 0.8187$$

Using $\alpha(z)$ and $\beta(z)$ thus determined, Y(z)/V(z) becomes as follows:

$$\frac{Y(z)}{V(z)} = \frac{F(z)B(z)}{F(z)B(z)H_1(z)} = \frac{1}{H_1(z)} = \frac{1}{z + 0.5}$$

Since V(z)/R(z) is

$$\frac{V(z)}{R(z)} = G_{model}\, H_1(z) = \frac{0.32}{z^2 - 1.2z + 0.52}\,(z + 0.5)$$

the pulse transfer function Y(z)/R(z) becomes

$$\frac{Y(z)}{R(z)} = \frac{Y(z)}{V(z)}\,\frac{V(z)}{R(z)} = \frac{0.32}{z^2 - 1.2z + 0.52} = G_{model}$$

The designed model-matching control system has the block diagram as shown in the next page.

Laplace Transform Pairs for Bilateral

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}(s) < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathrm{Re}(s) > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathrm{Re}(s) < 0$
6	$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -a$
7	$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < -a$
8	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	$\mathrm{Re}(s) > -a$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$\frac{1}{(s+a)^n}$	$\mathrm{Re}(s) < -a$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos w_0 t]u(t)$	$\frac{s}{s^2+w_0^2}$	$\mathrm{Re}(s) > 0$
12	$[\sin w_0 t]u(t)$	$\frac{w_0}{s^2+w_0^2}$	$\mathrm{Re}(s) > 0$
13	$[e^{-at}\cos w_0 t]u(t)$	$\frac{s+a}{(s+a)^2+w_0^2}$	$\mathrm{Re}(s) > -a$
14	$[e^{-at}\sin w_0 t]u(t)$	$\frac{w_0}{(s+a)^2+w_0^2}$	$\mathrm{Re}(s) > -a$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = u(t) * \cdots * u(t)$	$\frac{1}{s^n}$	$\mathrm{Re}(s) > 0$

Laplace Transform Property for Bilateral

Property	Signal	Laplace Transform	ROC
	$x(t)$, $x_1(t)$, $x_2(t)$	$X(s)$, $X_1(s)$, $X_2(s)$	R, R_1, R_2
Linearity	$ax_1(t)+bx_2(t)$	$aX_1(S)+bX_2(S)$	$R_1 \cap R_2$
Time Shifting	$x(t-t_0)$	$e^{-st_0}X(S)$	R
Shift in the s-Domain	$e^{s_0 t}x(t)$	$X(s-s_0)$	R
Time Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{s}{a})$	Scaled ROC
Conjugation	$x^*(t)$	$X^*(S^*)$	R
Convolution	$x_1(t)*x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the time-Domain	$\frac{dx(t)}{dt}$	$SX(S)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{dX(S)}{dS}$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{S}X(S)$	At least $R \cap \{\mathrm{Re}(s)>0\}$
Initial value theorem	$x(0^+)=\lim_{s\to\infty} SX(S)$		
Final value theorem	$\lim_{t\to\infty} x(t)=\lim_{s\to 0} SX(S)$		

Laplace Transform Property for Unilateral

Property	Signal	Laplace Transform
	$x(t)$, $x_1(t)$, $x_2(t)$	$\chi(s)$, $\chi_1(s)$, $\chi_2(s)$
Linearity	$ax_1(t)+bx_2(t)$	$a\chi_1(S)+b\chi_2(S)$
Shift in the s-Domain	$e^{s_0 t}x(t)$	$\chi(s-s_0)$
Time Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}\chi(\frac{s}{a})$
Conjugation	$x^*(t)$	$\chi^*(S)$
Convolution	$x_1(t)*x_2(t)$	$\chi_1(s)\chi_2(s)$
Differentiation in the time-Domain	$\frac{dx(t)}{dt}$	$S\chi(S)-x(0^-)$
	$\frac{d^2x(t)}{dt^2}$	$S^2\chi(S)-sx(0^-)-x'(0^-)$
Differentiation in the s-Domain	$-tx(t)$	$\frac{d\chi(S)}{dS}$

Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{S}\chi(S)$
Initial value theorem	$x(0^{+}) = \lim_{s\to\infty} S\chi(S)$	
Final value theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} S\chi(S)$	

Table of z transforms 1

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	—	—	Kronecker delta $\delta_0(k)$ 1, $k = 0$ 0, $k \neq 0$	1
2.	—	—	$\delta_0(n-k)$ 1, $n = k$ 0, $n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1-z^{-1}}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$
8.	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$1-e^{-akT}$	$\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
11.	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)e^{-akT}$	$\frac{1-(1+aT)e^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$

Table of z transforms 2

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
12.	$\frac{2}{(s+a)^3}$	$t^2 e^{-at}$	$(kT)^2 e^{-akT}$	$\frac{T^2 e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$	$akT-1+e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
18.			a^k	$\frac{1}{1-az^{-1}}$
19.			a^{k-1} $k=1,2,3,\ldots$	$\frac{z^{-1}}{1-az^{-1}}$
20.			ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21.			$k^2 a^{k-1}$	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.			$k^3 a^{k-1}$	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23.			$k^4 a^{k-1}$	$\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24.			$a^k \cos k\pi$	$\frac{1}{1+az^{-1}}$
25.			$\frac{k(k-1)}{2!}$	$\frac{z^{-2}}{(1-z^{-1})^3}$
26.			$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}$	$\frac{z^{-m+1}}{(1-z^{-1})^m}$
27.			$\frac{k(k-1)}{2!}a^{k-2}$	$\frac{z^{-2}}{(1-az^{-1})^3}$
28.			$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}a^{k-m+1}$	$\frac{z^{-m+1}}{(1-az^{-1})^m}$

Property table:

	$x(t)$ or $x(k)$	$\mathcal{Z}[x(t)]$ or $\mathcal{Z}[x(k)]$
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t) + bx_2(t)$	$aX_1(z) + bX_2(z)$
3.	$x(t+T)$ or $x(k+1)$	$zX(z) - zx(0)$
4.	$x(t+2T)$	$z^2X(z) - z^2x(0) - zx(T)$
5.	$x(k+2)$	$z^2X(z) - z^2x(0) - zx(1)$
6.	$x(t+kT)$	$z^kX(z) - z^kx(0) - z^{k-1}x(T) - \cdots - zx(kT-T)$
7.	$x(t-kT)$	$z^{-k}X(z)$
8.	$x(n+k)$	$z^kX(z) - z^kx(0) - z^{k-1}x(1) - \cdots - zx(k-1)$
9.	$x(n-k)$	$z^{-k}X(z)$
10.	$tx(t)$	$-Tz\frac{d}{dz}X(z)$
11.	$kx(k)$	$-z\frac{d}{dz}X(z)$
12.	$e^{-at}x(t)$	$X(ze^{aT})$
13.	$e^{-ak}x(k)$	$X(ze^{a})$
14.	$a^kx(k)$	$X\left(\frac{z}{a}\right)$
15.	$ka^kx(k)$	$-z\frac{d}{dz}X\left(\frac{z}{a}\right)$
16.	$x(0)$	$\lim_{z\to\infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim_{z\to 1} [(1-z^{-1})X(z)]$ if $(1-z^{-1})X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k) = x(k) - x(k-1)$	$(1-z^{-1})X(z)$
19.	$\Delta x(k) = x(k+1) - x(k)$	$(z-1)X(z) - zx(0)$
20.	$\sum_{k=0}^{n} x(k)$	$\frac{1}{1-z^{-1}}X(z)$
21.	$\frac{\partial}{\partial a}x(t,a)$	$\frac{\partial}{\partial a}X(z,a)$
22.	$k^mx(k)$	$\left(-z\frac{d}{dz}\right)^m X(z)$
23.	$\sum_{k=0}^{n} x(kT)y(nT-kT)$	$X(z)Y(z)$
24.	$\sum_{k=0}^{\infty} x(k)$	$X(1)$

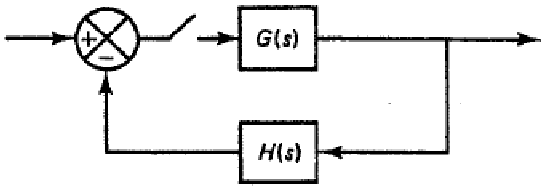
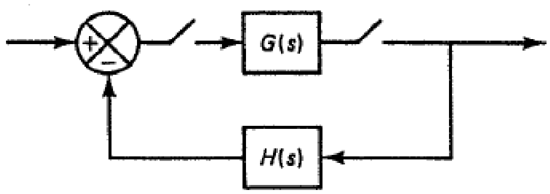
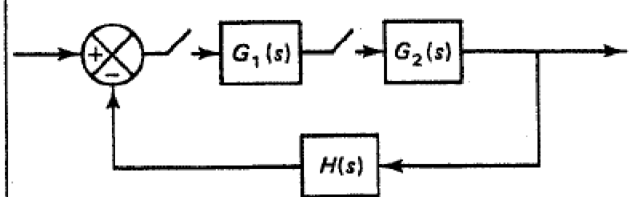
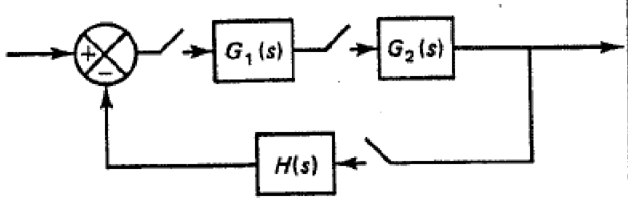
Table: z transform of $x(k+m)$ and $x(k-m)$

Discrete function	z Transform
$x(k+4)$	$z^4X(z) - z^4x(0) - z^3x(1) - z^2x(2) - zx(3)$
$x(k+3)$	$z^3X(z) - z^3x(0) - z^2x(1) - zx(2)$
$x(k+2)$	$z^2X(z) - z^2x(0) - zx(1)$
$x(k+1)$	$zX(z) - zx(0)$
$x(k)$	$X(z)$
$x(k-1)$	$z^{-1}X(z)$
$x(k-2)$	$z^{-2}X(z)$
$x(k-3)$	$z^{-3}X(z)$
$x(k-4)$	$z^{-4}X(z)$

Jury table is given as follows

Row	z^0	z^1	z^2	z^3	$\cdots$	z^{n-2}	z^{n-1}	z^n
1	a_n	a_{n-1}	a_{n-2}	a_{n-3}	$\cdots$	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3	$\cdots$	a_{n-2}	a_{n-1}	a_n
3	b_{n-1}	b_{n-2}	b_{n-3}	b_{n-4}	$\cdots$	b_1	b_0	
4	b_0	b_1	b_2	b_3	$\cdots$	b_{n-2}	b_{n-1}	
5	c_{n-2}	c_{n-3}	c_{n-4}	c_{n-5}	$\cdots$	c_0		
6	c_0	c_1	c_2	c_3	$\cdots$	c_{n-2}		
⋮	⋮							
$2n-5$	p_3	p_2	p_1	p_0				
$2n-4$	p_0	p_1	p_2	p_3				
$2n-3$	q_2	q_1	q_0					

Static error constants for typical closed-loop configurations of discrete time control system

Closed-loop configuration	Values of K_p, K_v, and K_a
	$K_p = \lim_{z \to 1} GH(z)$ $K_v = \lim_{z \to 1} \frac{(1 - z^{-1})GH(z)}{T}$ $K_a = \lim_{z \to 1} \frac{(1 - z^{-1})^2 GH(z)}{T^2}$
	$K_p = \lim_{z \to 1} G(z)H(z)$ $K_v = \lim_{z \to 1} \frac{(1 - z^{-1})G(z)H(z)}{T}$ $K_a = \lim_{z \to 1} \frac{(1 - z^{-1})^2 G(z)H(z)}{T^2}$
	$K_p = \lim_{z \to 1} G_1(z)HG_2(z)$ $K_v = \lim_{z \to 1} \frac{(1 - z^{-1})G_1(z)HG_2(z)}{T}$ $K_a = \lim_{z \to 1} \frac{(1 - z^{-1})^2 G_1(z)HG_2(z)}{T^2}$
	$K_p = \lim_{z \to 1} G_1(z)G_2(z)H(z)$ $K_v = \lim_{z \to 1} \frac{(1 - z^{-1})G_1(z)G_2(z)H(z)}{T}$ $K_a = \lim_{z \to 1} \frac{(1 - z^{-1})^2 G_1(z)G_2(z)H(z)}{T^2}$